Problèmes concrets d'analyse fonctionnelle. By P. Lévy. Paris, Gauthier-Villars, 1951. 14+484 pp. 4000 fr.

This work is a revised edition of Lévy's *Leçons d'analyse fonctionnelle* published in 1922. The purpose of the new title is to emphasize the difference in method and point of view between this book and the very abstract and general theories to which the name "functional analysis" is now commonly attached. Apart from the fact that Lévy restricts attention to certain special function spaces, there is a more striking difference between this work and most other publications on functional analysis, in that Lévy devotes most of his space to the study of functional differential equations. These are the analogues in function space of total differential equations, first order partial differential equations, and second order partial differential equations, especially the Laplace equation. The last type of problem (considered in Part III) involves considerable difficulties, even in choosing a suitable formulation for it. It is necessary to consider problems of measure in function space, the definition of mean value of a functional, and the proper definition of the Laplace operator. Lévy's procedure is to consider these ideas in n-dimensional space, and then to proceed to the limit as n tends to infinity, so as to obtain a mean value for certain functionals defined in L_2. When the method of passage to the limit is chosen so that certain requirements are satisfied, other aspects of the geometrical situation become rather bizarre. Lévy has extensively revised Part III, but he states in his Preface that he still does not regard it as in definitive form. Choice of another method of passage from finite-dimensional space to the space C of continuous functions leads to the Wiener integral, which has other applications in functional analysis (cf. Paley and Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 19, 1934, Chaps. 9 and 10; also later memoirs by R. H. Cameron and W. T. Martin). It seems reasonable to forecast that investigation will reveal still other methods of defining integrals in the spaces C and L_2, which will have interesting applications.

A noteworthy change from the first edition is the addition of Part IV, by Franco Pellegrino, on analytic functionals of analytic functions. About forty percent of Part I has been omitted, including the old Chapter 3 on Lebesgue and Stieltjes integrals, Chapter 7 on orthogonal functions, and Chapter 8 on the equations of Fredholm and Volterra and integrodifferential equations. For a review of the first edition by C. A. Fischer, see Bull. Amer. Math. Soc. vol. 29 (1923) p. 229.

Part I is entitled *The Foundations of the functional calculus.* The function spaces to be considered are described in Chapter 1. Principal emphasis is placed on the space L_2. Chapter 2 proceeds to discuss linear functionals and their representations, and the first variation of functionals. Chapters 3 and 4 are concerned with functionals of the second degree, and the second variation, and with functionals of arbitrary degree. Principal attention is paid to the case when these functionals have the special forms $\int_a^b k(t)x(t)dt$, $\int_a^b\int_a^b k(s, t)x(s)x(t)dsdt$, $\int_a^b k(t)x^2(t)dt$, etc.

Part II is almost unchanged, even to the numbering of the paragraphs. It is concerned with total "differential" equations in function space, with application of these results to the theory of the Green's function, with partial functional differential equations of the first order, and with applications of this last idea to set up a Hamilton-Jacobi theory for multiple integrals in the calculus of variations. It is not clear how this Hamilton-Jacobi theory provides any effective illumination of the difficulties inherent in the theory of extrema of multiple integrals. In connection with total differential equations in function space, it may be noted that such problems have been studied in abstract Banach space by Michal and Elconin (see Acta Math. vol. 68 (1937) pp. 71–107).

Part III is entitled *The notion of mean value in function space, and the generalized Laplace equation.* The approximating n-dimensional space in L_2 (for the interval $[0, 1]$) is composed of the functions constant on the interior of each of the intervals $((i-1)/n, i/n)$. When a complete orthogonal system of functions in L_2 is chosen so as to set up a correspondence with classical Hilbert space, another natural method of choosing an approximating n-dimensional space is suggested. The mean values of a functional given by these two methods of approximation need not be the same, even when both exist, and the mean value given by the Hilbert space method may depend on the order of the axes. Lévy gives conditions on the choice of the coordinate system to ensure that these mean values shall agree. Chapter 5 of this section, on harmonic functionals, discusses Green's formula, potentials of simple and of double layers and of volume distributions, the Dirichlet problem, and the problem of Plateau, in function space.

The fourth and concluding part of the book consists of six chapters —121 pages— on the subject of analytic functionals of analytic functions, written by Franco Pellegrino, a pupil of Luigi Fantappiè, who initiated this branch of the theory of functionals about 1925.

The functions admitted as arguments of analytic functionals are those called "biregular." Each biregular function $y_0(t)$ has a domain

M_0 which is open on the complex sphere, does not include the whole sphere, and need not be connected. A biregular function is locally analytic, and vanishes at ∞ if defined there. A "continuation" of a biregular function is not necessarily an analytic continuation. The class of biregular functions is made into a topological space $S^{(1)}$ by the system of neighborhoods (A, σ) of its points (y_0, M_0). For convenience we shall denote by $N(y_0, M_0; A, \sigma)$ the neighborhood (A, σ) of (y_0, M_0) consisting of all (y, M) with $M \supset A$ and $|y(t)-y_0(t)| < \sigma$ on A, where A is closed, $A \subset M_0$, $\sigma > 0$. With these neighborhoods $S^{(1)}$ is a T_0-space. A set consisting of a single function (y_0, M_0) has as points of accumulation all the restrictions of y_0 which lie in $S^{(1)}$. A linear set in $S^{(1)}$ is a collection $E = [(y, M)]$ such that every finite subcollection of the domains M has a non-null intersection, and every finite linear combination $\sum c_j y_j$ with complex coefficients lies in E. The domain of the sum $\sum c_j y_j$ is of course the intersection of the domains M_j. It is shown that a linear open set in $S^{(1)}$ consists of all the functions $y(t)$ which are biregular on a fixed closed proper subset A of the complex sphere. It is noteworthy that the space $S^{(1)}$ itself is not linear.

An analytic curve in $S^{(1)}$ is given by a parametric representation $y(t, \alpha)$, where for each α_0 in a set Ω open on the α-sphere, the corresponding function $y(t, \alpha_0)$ is biregular on its domain $M(\alpha_0)$ (which is open on the t-sphere), and for each t_0 in a set Ω' open in the t-sphere $y(t_0, \alpha)$ is regular in α, and where the complement $I(\alpha)$ of $M(\alpha)$ is continuous with respect to α in terms of the usual distance of point sets. A functional F defined on an open set R in $S^{(1)}$ is said to be locally analytic in case $F(y_1) = F(y_2)$ whenever y_2 is a continuation of y_1, and for every analytic curve $y(t, \alpha)$ lying in R, $F[y(\cdot, \alpha)]$ is a locally analytic function of α. The author states that it is unknown whether every analytic functional is continuous. However, an analytic functional which is bounded near a point of $S^{(1)}$ is necessarily continuous there.

A linear functional $F(y)$ is one which is defined and linear on a linear open set. It is continuous if and only if it is analytic. The class of linear analytic functionals is isomorphic to $S^{(1)}$, and in fact it follows from Cauchy's integral formula that every linear analytic functional F is expressible in the form

$$F(y) = \frac{1}{2\pi i} \int_C u(t) y(t) dt$$

where the path C encloses the singularities of u and excludes those of

y, and the function u is the "antisymmetric indicatrix" $u(\alpha) = F[1/(\alpha - t)]$.

Chapter IV is a very brief chapter devoted to "mixed functionals," i.e., to transformations on the space $S^{(1)}$. The operation D of differentiation is a linear analytic transformation, and linear transformations permutable with D are called "operators of the closed cycle" (cf. Volterra, *La teoria dei funzionali applicata ai fenomeni ereditaria*, Atti Congresso Int. di Bologna, vol. IV, 1928). The antisymmetric indicatrix

$$u(\alpha, z) = F\left(\frac{1}{\alpha - t}; z\right)$$

of such an operator $F(y; z)$ is a function of the difference $(\alpha - z)$. "Normal operators" are those having the functions t^n $(n = 0, 1, 2, \cdots)$ as characteristic vectors. The antisymmetric indicatrix of a normal operator is a function of z/α. The properties of normal operators are used to derive the theorem of Hadamard which relates the singularities of the functions defined by the power series $\sum a_n t^n$, $\sum b_n t^n$, and $\sum a_n b_n t^n$. However, little else is given in the direction of explicit applications of these concepts.

In Chapter V the author discusses the application of analytic linear functionals to the solution of linear differential equations, both ordinary and partial. The justification of the symbolic calculus is taken up first. The later sections outline five methods for the explicit solution of certain classes of linear partial differential equations.

Chapter VI takes up the first and higher variations of nonlinear functionals, and defines the first derivative, for example, as the antisymmetric indicatrix of the first variation. These notions lead to an analogue of the Taylor expansion, called "the series of Fantappiè." Brief reference is made to maxima and minima of real analytic functionals.

Most of the details are omitted in the later chapters. References are included to the work of various authors, listed in the bibliography of 119 items, 49 of which are by Fantappiè. Unfortunately no page numbers are given in many of the references. Mention is made of a forthcoming work by Fantappiè and Pellegrino, which is to contain a more detailed exposition of the theory.

The theory of Fantappiè is interesting in being an explicit development of functional analysis for a case where the basic function space is not a Banach space. It may throw some light on the form which an abstract theory of analytic functionals should take.

L. M. GRAVES