THE JUNE MEETING IN EUGENE

The four hundred eight-second meeting of the American Mathematical Society was held at the University of Oregon, Eugene, Oregon, on Saturday, June 21, 1952. A total of 173 persons registered for the meeting, including the following 86 members of the Society:

C. B. Allendoerfer, N. C. Ankeny, T. M. Apostol, B. H. Arnold, M. G. Arsove, G. A. Baker, R. W. Ball, C. R. Ballantine, J. P. Ballantine, R. A. Beaumont, R. F. Bell, J. S. Biggerstaff, Z. W. Birnbaum, Gertrude Blanch, R. N. Bradt, J. V. Breakwell, J. L. Brenner, B. W. Brewer, F. H. Brownell, D. G. Chapman, Harold Chatland, Paul Civin, B. H. Colvin, K. L. Cooke, C. M. Cramlet, E. L. Crow, G. B. Dantzig, D. B. Dekker, Douglas Derry, W. J. Dixon, Melvin Dresher, L. A. Dye, E. A. Fay, G. E. Forsythe, J. W. Gaddum, R. S. Gardner, K. S. Ghent, M. A. Girshick, E. G. Goman, C. H. Gordon, F. L. Griffin, Margaret Gurney, S. G. Hacker, M. E. Haller, A. J. Hoffman, V. E. Hoggatt, Mark Kac, J. M. Kingston, M. S. Knebelman, L. M. LeCam, D. H. Lehmer, R. B. Leipnik, J. C. R. Li, A. E. Livingston, Eugene Lukacs, M. W. Maxfield, A. S. Merrill, W. E. Milne, T. S. Motzkin, A. F. Moursund, W. L. Nicholson, O. M. Nikodým, Ivan Niven, A. R. Noble, T. G. Ostrom, D. B. Owen, T. S. Peterson, J. H. Raymond, P. V. Reichelderfer, R. A. Rosenbaum, Herman Rubin, W. G. Scobert, R. G. Selfridge, W. L. Shepherd, Judson C. Smith, Andrew Sobczyk, R. D. Stalley, W. M. Stone, Gabor Szegö, F. H. Tingey, G. E. Uhrich, J. L. Ullman, J. G. Wendel, R. F. Williams, R. J. Wisner, F. H. Young.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Dr. G. E. Forsythe, National Bureau of Standards, Los Angeles, California, delivered an address entitled *Solving linear equations is not trivial*. Dr. Forsythe was introduced by Professor W. E. Milne. There were sessions for contributed papers in the morning and afternoon, presided over by Professors Gabor Szegö, R. A. Beaumont, and R. A. Rosenbaum.

Following are abstracts of papers presented at the meeting. Papers with abstract numbers followed by "*t*" were presented by title. Paper number 505 was presented by Professor Apostol, number 518 by Dr. Wendel, number 523 by Dr. M. W. Maxfield, number 531 by Professor Civin, and number 556 by Professor Reichelderfer. Dr. J. E. Maxfield was introduced by Dr. M. W. Maxfield and Professor Hostetter by Professor W. T. Puckett.

ALGEBRA AND THEORY OF NUMBERS

505. D. R. Anderson and T. M. Apostol: *The evaluation of Ramanujan's sum and generalizations.*

Ramanujan obtained an expression for the sum $R(n; k)$ of the nth powers of the primitive kth roots of unity in terms of the Möbius function $\mu(n)$, namely, $R(n; k) = \sum_{d|(n,k)} d\mu(k/d)$, where d runs over the divisors of the g.c.d. (n, k). The authors reduce this sum to a closed form, $R(n;\ k)=c\mu(b)\phi(a)/\phi(c)$, where $\phi(n)$ is Euler's totient, $a=(n,\ k)$, $b=k/a$, and $c=(a,\ b)$. Similar formulae are obtained for sums of the form $\sum_{d|(n,k)} f(d)g(k/d)\mu(k/d)$, where $g(n)$ is multiplicative and $f(n)$ is completely multiplicative. Also, a connection is established between these generalized sums and certain exponential sums involving nth powers of kth roots of unity. (Received May 5, 1952.)

506. N. C. Ankeny: *An application of algebraic geometry to number theory.*

Let $f(X)$ be a polynomial whose coefficients are rational integers. Define $N(f(X),\ p)$ as the number of different values $f(i)$ (mod p) for $i=0,\ 1,\ \cdots,\ p-1$ with p a prime number. The author then finds necessary and sufficient conditions on $f(X)$ such that $\lim_{p\to\infty} N(f(X),\ p)/p$ exists, and further shows that if the limit exists it depends only upon the degree of $f(X)$. The method is based on the ideas of A. Weil on algebraic curves over finite fields. Some applications are given. (Received May 5, 1952.)

507*t*. H. W. Becker: *Eels vectors.*

W. C. Eels, Amer. Math. Monthly vol. 21 (1914) pp. 269–273, gave an extensive table of solutions of the Diophantine system: $(t)\ x^2+y^2+z^2=t^2$, $(u)\ x^2+y^2=u^2$. Taking x and t odd, from the solution (11) of (t), R. D. Carmichael, *Diophantine analysis*, (1915) p. 38, is derived the Eels vector of the first kind: $t,\ u=(M^2+N^2+P^2+Q^2)^2 \pm 4(MP+NQ)^2=T^2\pm Z^2$, $x=(M^2-N^2-P^2+Q^2)^2-4(MN-PQ)^2=X^2-Y^2$, $y=4(M^2-N^2-P^2+Q^2)(MN-PQ)=2XY$, $z=4(M^2+N^2+P^2+Q^2)(MP+NQ)=2TZ$, in which both component Pythagorean triangles are primitive. Another primitive solution, in which however the component triangles are not necessarily primitive, has $m=p=rs$, $q=r^2-s^2-n$ in (11). The case $q=0$ gives $t,\ u=r^4\pm s^4$, $x=t-z=(r^2-s^2)^2$, $y=2rs(r^2-s^2)$, $z=2r^2s^2$. E.V. of the second kind, (t), $(v)\ y^2+z^2=v^2$ appear on solution of $(m^2+q^2)(n^2+p^2)=\square$ in (11). There are no E.V. of the second kind with three of $m,\ n,\ p,\ q$ odd: this is Spunar's theorem, Amer. Math. Monthly vol. 24 (1917) p. 393, proven under the impression that it was actually a proof of the impossibility of Martin vectors meeting the conditions (t), (u), (v), and $(w)\ x^2+z^2=w^2$. E.V. of both kinds (on reduction) are corollaries of (9), Carmichael, ibid.: $t,\ x=(M^2+Q^2R^2)^2\pm R^2(N^2+P^2)^2$, $y=4R(MN-PQR)(MP+NQR)$, $z=2R[(P^2-N^2)(M^2-Q^2R^2)+4MNPQR]$, $v=2R(P^2+N^2)(M^2+Q^2R^2)$. If further $x\mp y=u\mp z$, this is Frenicle's problem, Dickson's *History*, vol. II, pp. 184–186. (Received May 8, 1952.)

508t. H. W. Becker: *Williams vectors.*

In his *Algebra*, 1832, p. 419, J. D. Williams challenged his contemporaries with the Diophantine simultaneous equations (t), (u), (v) above, and gave the probably simplest solution $u,\ v,\ x,\ y,\ z,\ t\ =185,\ 680,\ 153,\ 104,\ 672,\ 697$. This was erroneously stated to be erroneous by Zerr, Amer. Math. Monthly vol. 15 (1908) p. 17, Dickson's *History*, vol. II, p. 492 (despite Cunliffe, 452, and Ward, 506). The system reduces to $\mu\nu=(i^2+j^2)(k^2+l^2)=(f^2h^2+e^2g^2)(e^2h^2+f^2g^2)=t^2$, $2(ijkl)^{1/2}=2efgh=y$, for which 15 types of solution are considered, $\xi\neq\square\neq1$: $(0.1)\mu=\square$, $\nu=\square$; (0.2) $\mu=\xi\square$, $\nu=\xi\square$; (0.3) $\nu=\mu\square$; (0.4) $\mu=\nu$; and (I.1), etc., according as 1, 2, or 4 of $i,\ j,\ k,\ l$ are squares. Many primitive parametric and numerical solutions are tabulated, including 16 with $t<10^4$, such as: (0.1) 975, 952, 495, 840, 448, 1073; (0.2) 1073, 561, 952, 495, 264, 1105; (0.3) 925, 3444, 533, 756, 3360, 3485; (I.3) 2175, 2431, 1092, 1881, 1540, 2665; (II.2) 533, 765, 520, 117, 756, 925; (II.3) 1073, 520, 975, 448, 264, 1105; (IV.3) Williams' example. Valuable preliminaries to or paraphrases of the problem are in Dickson, ibid., pp. 406–407, 428, 505–507, 630–634, 644–646, 667–668, Carmichael, ibid., chap. IV. The system has at least 5 different geometric interpretations, as the obvious parallelopipedon of volume xyz, and Hero triangle with sides u^2, v^2, x^2+z^2 and area $xyzt$, Turriere, L'Enseignment Mathématiques vol. 18 (1916) Arithmogeometrie. (Received May 8, 1952.)

509*t*. H. W. Becker: *Petrus and Carmichael vectors.*

Williams' problem was revived by R. D. Carmichael, *Diophantine analysis*, p. 23, ex. 8. His triadic solution (unpublished) takes off from 24–25 and ex. 6, ibid. (Hillyer), and is more general than the included Euler triad, Dickson's *History*, vol. II, p. 474. All solutions of $y/4=(abcd(a^2-b^2)(c^2-d^2))^{1/2}=(a, b, c, d)$, $t=(a^2+b^2)(c^2+d^2)$, $u, x=4abcd\pm(a^2-b^2)(c^2-d^2)$, $v, z=2(ac\mp bd)(ad\pm bc)$ are referred to as Carmichael vectors; they are type (.1) W.V. The first solution is (5, 2, 6, 1). Many explicit parametric solutions are developed from cognate relations in Dickson, ibid., pp. 172–176, 449, 459–474, 493, 511, 527, 661. There is a triadization transform, $T[(a, b, c, d), (a, b, c^*, d^*)]=(c, d, c^*, d^*)$, the latter often a maverick numerical solution yet to be generalized. There is a congruent number transform, $Q(a, b, c, d)=((a^2+b^2)^2, 4ab(a^2-b^2), (c^2+d^2)^2, 4cd(c^2-d^2))$, where Y^2 has 7 square factors, the maximum possible. Petrus (1674), Dickson, ibid., pp. 446–450, 2, 4, 6, 8, demanded 3 numbers such that their 6 sums and differences are squares. The solution is a subclass of C.V. Petrus gave a transform P by which every P.V. $=P$(W.V.). Remarkably, $P^2=Q$. The inverse Petrus transform: $((T\pm U)/2)^{1/2}=t, y$, $((T\pm X)/2)^{1/2}=u, v$, $((U\pm X)/2)^{1/2}=x, z$, correlates known, or yields new, W.V.; e.g., Euler's P.V., Dickson, ibid., p. 450, with a misprint which should read $q=-(f^4-g^4)^2-16f^4g^4$. (Received May 8, 1952.)

510*t*. H. W. Becker: *Pythagorean tetrahedrons, hexahedrons, and graphs.* Preliminary report.

Dickson's *History*, vol. II, pp. 221–224, reports tetrahedrons with rational faces and volume, yet none whose four faces are Pythagorean triangles. But the six components of a W.V. can be rearranged to form such a Pythagorean tetrahedron, of integer volume $xyz/6$, with two right angles at each of two vertices. From this point of view, each W.V. or P.T. depends on $(i^2-\phi^2)(k^2-\lambda^2)=(\zeta^2\theta^2-\epsilon^2\eta^2)$. $(\epsilon^2\theta^2-\zeta^2\eta^2)=x^2$, $2(i\phi k\lambda)^{1/2}=2\epsilon\zeta\eta\theta=z$ in two ways. So, automatically appear many explicit parametric solutions of these forms, for general or square values of the Greek letters, numerical solutions of which were supplied by Euler and Gerardin, Dickson, ibid., pp. 448, 458, 505, 661–662. Every Petrus vector is of types (− .1), (−IV). Any two P.T. with a common face make a Pythagorean hexahedron. Every P.T. u, v, x, y, z, t transforms to four others: ty, vx, vy, xy, xz, uv; uz, ty, xz, yz, uy, uv; uv, tz, ty, xz, vz, tv; tx, uv, ux, zx, ty, tu, each having three faces similar to three faces of the P.T. transformed. Thus from the first C.V. is derived the first P.H., whose base triangle is 1105, 1073, 264, with edges 448, 520, 975, and 495, 561, 952 at the apices. But the regular graph of ten branches, the pentagon-pentacle, all of whose ten triangles are Pythagorean, is impossible. In general, what graphs admit of Pythagorean triangulation? (Received May 8, 1952.)

511*t*. Leonard Carlitz: *Congruences connected with three-line latin rectangles.*

Let $K_n=K(3, n)$, the number of reduced three-line latin rectangles. Riordan

(Amer. Math. Monthly vol. 59 (1952) pp. 159–162) has proved the congruences $k_{n+p} \equiv 2k_n$, $K_{n+p} \equiv 2K_n \pmod{p}$, where p is a prime >2 and k_n is related to K_n by means of a certain recurrence. In the present paper it is proved that for arbitrary m, $k_{n+m} \equiv 2^m k_n$, $K_{n+m} \equiv 2^m K_n \pmod{m}$. Indeed if we put $\Delta f(n) = f(n+m) - 2^m f(n)$, $\Delta^s f(n) = \Delta\Delta^{s-1} f(n)$, then we have $\Delta^r k_n \equiv \Delta^r K_n \equiv 0 \pmod{m^r}$ for $r \geqq 1$. (Received April 17, 1952.)

512*t*. Leonard Carlitz: *Congruences for the ménage polynomials.*

Put $U_n = U_n(t) = \sum_0^n u_{n,r} t^r$, where $u_{n,r}$ denotes the generalized ménage number. Riordan (Duke Math. J. vol. 19 (1952) pp. 27–30) has proved the congruence $U_{p^2+n} \equiv (t^{p^2} - 1) U_n \pmod{p}$, where p is a prime >2. In the present paper it is shown that $U_{m^2+n} \equiv (t-1)^{m^2} U_n \pmod{m}$, where m is an arbitrary integer. (Received April 17, 1952.)

513*t*. Leonard Carlitz: *Some congruences for Bernoulli numbers of higher order.*

S. Wachs (Bull. Sci. Math. (2) vol. 71 (1947) pp. 219–232) proved $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2}$ for p prime $\geqq 3$. The writer (Pacific Journal of Mathematics vol. 2, no. 2 (1952)) improved this to $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3}$ for $p > 3$. In the present paper the congruence $B_{p+2}^{(p+1)} \equiv p^{3/16} \pmod{p^4}$ for $p > 3$ is obtained. The writer had also proved $B_p^{(p)} \equiv p^2/2 \pmod{p^3}$. This is improved to $B_p^{(p)} \equiv -p^2(p-1)!/2 \pmod{p^5}$ for $p > 3$. (Received April 21, 1952.)

514*t*. Leonard Carlitz: *Some special equations in a finite field.*

In the first part of the paper asymptotic formulas are obtained for the number of solutions of equations of the type $f_1(\zeta_1) + \cdots + f_r(\zeta_r) = \alpha$, where the f_i are polynomials with coefficients in $GF(q)$; use is made of some theorems of Mordell. In the second part exact formulas are obtained for the number of solutions of equations of the form $Q(\zeta_1, \cdots, \zeta_r) = f(\eta_1, \cdots, \eta_s)$, where Q denotes a quadratic form while f is an arbitrary polynomial satisfying certain restrictions. In the latter problem the ζ_i are also permitted to be polynomials in $GF[q, x]$ of degree $<m$. (Received March 18, 1952.)

515*t*. Leonard Carlitz: *Weighted quadratic partitions over a finite field.*

This note is concerned with the evaluation of sums of the type $S = \sum e(2\lambda_1\zeta_1 + \cdots + 2\lambda_r\zeta_r)$ extended over all $\zeta_i \in GF(q)$ such that $Q(\zeta_1, \cdots, \zeta_r) = \alpha$, where $\alpha, \lambda_i \in GF(q)$ and $e(\alpha) = e^{2\pi i t(\alpha)/p}$, $t(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}}$, $q = p^n$, $p > 2$. The evaluation of S depends upon the Kloosterman sums $\sum_\zeta e(\alpha\zeta + \beta\zeta^{-1})$, and $\sum_\zeta \psi(\zeta) e(\alpha\zeta + \beta\zeta^{-1})$, where $\psi(\zeta) = +1$ or -1 according as ζ is a square or a nonsquare of the field. (Received March 18, 1952.)

516*t*. C. W. Curtis: *A note on noncommutative polynomials.*

An integral domain R (i.e. an associative ring without divisors of zero) is said to have property (M) in case any two nonzero elements of R have a nonzero common right multiple. Property (M) is necessary and sufficient that R can be imbedded in a right quotient division ring. A ring S containing R is an *extension of type O* of R if S contains an element x not in R such that each element of S can be expressed uniquely in the form $\sum_{i=0}^{m} x^i r_i$, $r_i \in R$, where $rx = xT(r) + D(r)$, $T(r)$, $D(r) \in R$. In this note it is proved that if S is a ring containing an integral domain R with property (M), such that S is the join of a well ordered sequence of subrings beginning with R, where each subring is an extension of type O of the preceding, then S is an integral domain with property (M). In the case of a single extension of type O of a division ring, this result is due to Ore (Ann. of Math. vol. 34 (1933) pp. 480–508). A consequence of the result is that the general enveloping algebra of a solvable Lie algebra over a field of characteristic zero has property (M). (Received April 28, 1952.)

517t. H. A. Dye: *On group representations of finite type.*

It is proved that a connected locally compact group G having a faithful strongly continuous unitary representation in a finite W^*-algebra M is the direct product of a vector group and a compact group. The group G will be separable if M is countably decomposable. Applied to connected Lie groups, this result subsumes a theorem of Segal and von Neumann on unitary representations of semi-simple groups (Ann. of Math. vol. 52 (1951) pp. 509–517) and yields the following analogue in the solvable case: the kernel of a strongly continuous unitary representation of a connected solvable group in a finite ring contains the commutator subgroup. Another corollary concerns the dimension-type of group representations: factors of type II_1 do not arise among the W^*-algebras enveloping strongly continuous unitary representations of connected locally compact groups. Similar results have been conjectured and proved independently by R. V. Kadison and I. M. Singer. (Received May 5, 1952.)

518. Melvin Hausner and J. G. Wendel: *Ordered vector spaces.*

An *ordered vector space* is a vector space V over the reals, simply ordered under a relation $>$ and satisfying the usual axioms. Examples are furnished by spaces V_T defined as follows: T is a simply ordered set, V_T is the linear space of those real functions f which take nonzero values on well-ordered subsets of T, with $f>0$ meaning that $f(t_0)>0$, where t_0 is the least $t \in T$ at which $f(t) \neq 0$. A subspace of V_T is called *large* if it contains the characteristic functions of the points of T, and the truncations of its elements, g being a truncation of f in case for some t_0, $g(t) = f(t)$ or 0 according as $t < t_0$ or $t \geqq t_0$. *Theorem: The ordered vector space V is isomorphic to a large subspace of a uniquely determined V_T.* The chief tool in the proof is a notion of equivalence in the set of positive elements of V: $x \sim y$ means that $ax < y < bx$ for some real a, $b > 0$. The equivalence classes $[x]$ are simply ordered by the rule $[x] < [y]$ if and only if $x > cy$, all $c > 0$. T is the set of all equivalence classes, with this ordering. The construction of the isomorphism of V into V_T is begun by mapping an arbitrary system of representatives of the equivalence classes onto the characteristic functions of the respective points of T, and is completed by transfinite induction. (Received June 15, 1952.)

519. D. H. Lehmer: *Certain character matrices.*

Let $\chi(n)$ be a real nonprincipal character modulo the prime p (Legendre symbol). The paper is concerned with the square matrices M_α ($\alpha = 0, 1, \cdots, p-1$) of order $p-1$ whose general element $a_{ij} = \chi(\alpha + i + j)$. For each M_α it is possible to give explicit formulas for its determinant, its inverse, its characteristic roots, two of its latent vectors, and even the general element of its kth power, the latter depending in a peculiar way on the Fibonacci numbers in case $\alpha \neq 0$. Such matrices are useful in testing the correctness and efficacy of different matrix methods. (Received May 9, 1952.)

520*t*. O. I. Litoff: *On the equality of three subgroups of the general linear group.* Preliminary report.

Let $GL_n(R)$ be the group of nonsingular $n\times n$ matrices with elements in R. Let $U_n(R)$ be the subgroup of all elements of determinant 1. Let T be the group generated by the transvections (J. Dieudonné, Bull. Soc. Math. France vol. 71 (1943) pp. 27–45). It is known that if R is a division ring (Dieudonné, loc. cit.) or the integers mod p^r (J. Brenner, Ann. of Math. vol. 39 (1938) pp. 472–493), then $T=U$ and (except when $n=2$ and 2 is not invertible) $T=C$ (commutator subgroup). This paper extends these results to the case in which R is a commutative ring with unit which is either Euclidean or a valuation ring. Recently Hua and Reiner (Trans. Amer. Math. Soc. vol. 71 (1951) pp. 331–348) have given a different proof that $U=C$ in the case of the integers. (Received May 8, 1952.)

521*t*. O. I. Litoff: *On the normal subgroups of the general linear group.* Preliminary report.

This paper generalizes a classical theorem on the normal subgroups of the general linear group over a field (for example, J. Dieudonné, Bull. Soc. Math. France vol. 71 (1943) pp. 27–45) and a result of Brenner (Ann. of Math. vol. 39 (1938) pp. 472–493) for the integers mod p^r. Let R be a commutative valuation ring with unit which may have divisors of zero. (If $n=2$, it is further assumed that 2 is invertible and that the residue class field of R is not the field of three elements.) For any ideal I in R let N_I^* be the (normal) subgroup of all A in $GL_n(R)$ such that $A\in$center (mod I) and N_I be the (normal) subgroup of all A in $U_n(R)$ such that $A=1$ (mod I). For any normal subgroup N, let J be the smallest ideal such that $N\subset N_J^*$. Then $N_J\subset N\subset N_J^*$. (Received May 8, 1952.)

522. J. E. Maxfield: *Normal k-tuples.*

The definition of a normal k-tuple to the scale r is analogous to the definition of a normal number to the scale r. By use of a theorem stating that a k-tuple is normal if and only if a certain number derived from the k-tuple is normal and by use of the concept of k-dimensional uniform distribution (mod 1), most of the known results on normal numbers are extended to k-tuples. Further, it is shown that if A is a regular k by k matrix with rational elements and β is a normal k-tuple, then $\gamma=\beta A$ is a normal k-tuple. Let P be any permutation of the digits $0, 1, \cdots, r-1$, and let $P\alpha$ be the number obtained from α by performing the permutation on all the digits of α. If $\beta=(\alpha_1, \alpha_2, \cdots, \alpha_k)$ is normal to the scale r, then so are: (1) $(\alpha_1, \cdots, \alpha_{i-1}, P\alpha_i, \alpha_{i+1}, \cdots, \alpha_k)$ and (2) the k-tuple obtained from β by the application of P to the k-digits $b_m, b_{2m}, b_{3m}, \cdots$, where the mth k-digit is the k-tuple composed of the mth digits of the α_i, to the scale r. (Received May 2, 1952.)

523. J. E. Maxfield and Margaret W. Maxfield: *The existence of primitive roots* (mod p^r) *that are less than* p.

It is proved that for any odd prime p there is a primitive root (mod p^r) between 0 and p. This result is a corollary to the theorem of the paper, which applies to any divisor $d>1$ of $\phi(p)$ as exponent rather than to $\phi(p)$ alone. With the problem broken into cases according to whether d is not divisible by 2, divisible by 2 once, or divisible by 2 more than once, it is shown that an integer b can be found whose period (mod p) is d but whose period (mod p^2) is pd. An inductive argument extends the result to the modulus p^r. (Received April 28, 1952.)

524*t*. M. J. Norris: *Cofinally concentrated directed systems.*

A directed system is said to be cofinally concentrated if the cardinal of a set of elements not after a given arbitrary element is always less than the cardinal of the system. The main result is that every cofinally concentrated directed system has a well-ordered cofinal subset. (Received May 5, 1952.)

525*t*. W. L. Parker and B. A. Bernstein: *On uniquely solvable Boolean equations.*

Whitehead obtained a criterion for the unique solvability of a Boolean equation. The criterion is convenient when the equation is expressed in terms of logical addition, multiplication, and negation. The authors of the present paper obtain additional convenient criteria for the unique solvability of a Boolean equation expressed in terms of the logical operations, as well as suitable criteria for the unique solvability of any Boolean equation or system of Boolean equations expressed in terms of the ring operations of symmetric difference and multiplication. For each type of uniquely solvable equation or system of equations the solution is given in a form appropriate to the type. In the discussion of the Boolean ring equation a *normal ring form* of the equation is used and a simple relation between this form and the normal logical form is obtained. Systems of Boolean ring equations, linear and nonlinear, are studied with the help of "*complexes*"; systems of n linear equations in n unknowns are also studied with the help of ring determinants. In an incidental note a criterion is obtained for determining if the roots of one of two Boolean equations are also roots of the other. (Received April 28, 1952.)

526*t*. Dov Tamari: *On one-sided linear homogeneous equations in general rings.*

A (linear) equation with the unknowns on the right of the coefficients is a *right* equation, and a system of right equations is a *right* system $\mathfrak{S}_r$. *Solvable* means *nontrivially* solvable in the ring of coefficients R, and similarly *solution* means a nontrivial solution. R may contain divisors of zero and is said to have the property P_r if for any two $a_i \in R$, $a_i \neq 0$, there exists $b_i \in R$ such that $a_1 b_1 = a_2 b_2 \neq 0$: existence of C.R.M.$\neq 0$. One can extend to such rings the theorem, classical in fields: A system of n right linear homogeneous equations in $n+1$ or more unknowns is solvable. The straightforward proof, essentially different from the usual elimination process which can not be applied, is simpler than the classical one for fields. It is based on *shortest solutions*, i.e. solutions having the least number >0 of nonzero components. If R is not P_r but P_1, elimination can be applied to $\mathfrak{S}_r$. But as (generally) $\mathfrak{S}_r$ has no solution, this does only transform the *Diophantian problem* and is more useful to decide the eventual unsolvability of a specified system. (Received May 8, 1952.)

527*t*. Dov Tamari: *On the embedding of general Lie rings in quotient fields.*

Let $R[X]$ be a *locally finite* ring (not necessarily an algebra in the narrow sense) of Lie polynomials, i.e.: R any ring; X a finite or infinite set of symbols; $ax = xa$ $(a \in R,\ x \in X)$; $[x_i, x_j] \equiv x_i x_j - x_j x_i = \sum \left[{}^{i,j}_{k}\right] x_k \left(\left[{}^{i,j}_{k}\right] \in R\right)$, and any *finite* subset $Y \subset X$ contained in a *finite* subset $\overline{Y}$ such that $R[\overline{Y}]$ is a *Lie* (*sub*)*ring*. For any *linear order* $\mathfrak{O}$ of X every element of $R[X]$ has at least one $\mathfrak{O}$-*standard form* $\sum a_{\cdots n_i \cdots n_j \cdots} \cdots x_i^{n_i} \cdots x_j^{n_j} \cdots$ $(x_i < x_j$ in $\mathfrak{O})$, and $R[X]$ is represented *on* the set $R(X)$ of ordinary (commutative) polynomials. The construction of *formally nontrivial* C.R.(L.)Ms. in $R[X]$ reduces to solving a system of right (left) linear homogenous equations in R (see preceding abstract). For sufficiently high degrees of the unknown standard forms the number of unknowns exceeds that of equations. If the fundamental theorem on systems of right (left) linear homogeneous equations is valid in R, any two $P, Q \in R[X]$ of degree $\leqq m$ with $x_i \in Y$ have at least C.R.(L.)Ms. of any degree $r \geqq c_v m$ where $c_v = 1/(1-2^{-1/v})$ and $v = \text{card } \overline{Y}$ a natural number. If the representation by standard forms is *unique*, $R[X]$ is a *proper* Lie ring, and *formally nontrivial* is always *nontrivial*; with R also $R[X]$ is regular (r., l.). This shows the embeddability of the Birkhoff-Witt algebras in fields of quotients by the well known construction of Ore. (Received May 8, 1952.)

ANALYSIS

528. M. G. Arsove: *Functions of potential type.*

A function w representable as the difference of two subharmonic functions is called *δ-subharmonic*. With the introduction of the characteristic function $T_r(w)$ (*Functions representable as differences of subharmonic functions*, Proceedings of the International Congress of Mathematicians, Cambridge, 1950) the concept of *order* and the theory of functions of finite order (familiar for entire meromorphic functions) extend to entire δ-subharmonic functions. A complete entire δ-subharmonic function is said to be of *potential type* provided it has order zero and its mass distribution has finite total variation; we have as a criterion that $\lim_{r\to 0} [T_r(w)/\log r]$ exist finitely. Let E be a bounded nonpolar Borel set and q the equilibrium distribution of the unit mass on E. Under the norm $\|w\| = |\int_E w dq| + \lim_{r\to 0} [T_r(w)/\log r]$ functions of potential type form a Banach space, and thereby play the fundamental role of completing the space of potentials. Moreover, a number of further problems that arise naturally in potential theory can be solved only in terms of functions of potential type. (Received May 5, 1952.)

529t. R. P. Boas: *Entire functions with negative zeros.*

A theorem of Valiron [Ann. Fac. Sci. Univ. Toulouse (3) vol. 5 (1914) pp. 117–257] states that for an entire function $f(z)$ of order less than 1 with real negative zeros the conditions (1) $\log f(r) \sim \pi \csc \pi\rho r^\rho$, $r \to \infty$, and (2) $n(r) \sim r^\rho$ are equivalent; here $n(r)$ is the number of zeros of absolute value not exceeding r. For recent simplified proofs see Bowen [Quart. J. Math. Oxford Ser. vol. 19 (1948) pp. 90–100], M. Heins [Ann. of Math. (2) vol. 49 (1948) pp. 200–213]. Paley and Wiener [*Fourier transforms in the complex domain*, New York, 1934, p. 70] proved a theorem for functions of order 1 which is equivalent to saying that when $\rho = 1/2$ and $f(0) = 1$, (1) and (2) are equivalent to $\int_0^\infty x^{-3/2} \log|f(-x)| dx = -\pi^2$. The author shows that for $0 < \rho < 1$ and $0 < \sigma < 1$, (1) and (2) are equivalent to $\int_0^r x^{-1-\sigma} \{\log|f(-x)| - \pi \cot \pi\sigma n(x)\} dx \sim \pi(\rho-\sigma)^{-1} \cdot (\cot \pi\rho - \cot \pi\sigma) r^{\rho-\sigma}$. (Received May 2, 1952.)

530t. R. C. Buck: *On a theorem of Pólya.*

A new and very simple proof is obtained for the following well known theorem; let $f(z)$ be entire, of order one and zero type, and let $f(n) = O(n^r)$ for $n = 0, \pm 1, \pm 2, \cdots$; then $f(z)$ is a polynomial of degree r at most. The proof depends on the use of the function $G(z) = \sum_0^\infty \Delta^n f(0) z^n$. After a preliminary reduction, this is shown to be entire and bounded in the half planes $x \geqq -1/2$ and $x \leqq -1/2$, so that $G(z)$ is constant, and from this $f(z)$ is a polynomial. (Received May 5, 1952).

531. Paul Civin and H. E. Chrestenson: *The multiplicity of a class of perfect sets.*

Let the perfect set P be formed in the following way: Let $[0, 2\pi] = p_1^0$. At the mth stage of the construction remove the open interval d_i^{m+1} from p_i^m and call the remaining intervals p_{2i-1}^{m+1}, p_{2i}^{m+1}, $i=1, \cdots, 2^m$. Let $\epsilon_m = \sup_{i=1,\cdots,2^{m-1}} (d_i^m / p_i^{m-1})$, and $\theta_m = \sup_{i=1,\cdots 2^m} (p_{2i-1}^{m+1}/p_{2i}^{m+1}, p_{2i}^{m+1}/p_{2i-1}^{m+1})$. N. K. Bary [Fund. Math. vol. 9 (1928) pp. 84–100] showed that the hypotheses (1) $\epsilon_m = o(1)$ and (2) $\theta_m = O(1)$ implied that P was a set of multiplicity for trigonometrical series. She also conjectured that (2) was superfluous. S. Verblunsky [Acta Math. vol. 65 (1935) pp. 291–305] introduced a lemma to establish the Bary conjecture. The authors indicate the falsity of the Verblunsky lemma and prove that the multiplicity follows in the intermediate case in which (2) is replaced by $\theta_m = o(1/\sup_{m \leq n} \epsilon_n)$. (Received May 2, 1952.)

532. K. L. Cooke: *The asymptotic behavior of the solutions of linear and nonlinear differential-difference equations.*

Consider the equations (1) $u'(t+1) = a(t)u(t) + b(t)u(t+1)$ and (2) $u'(t+1) = a(t)u(t) + b(t)u(t+1) + D(u(t), u(t+1))$, where $D = \sum_{i+j \geq 2} b_{ij}(t)u(t)^i u(t+1)^j$. $a(t)$, $b(t)$, and the $b_{ij}(t)$ are assumed to be real and continuous for $t > t_0$. It is also supposed that $|b_{ij}(t)| \leq b_{ij}$ and that the series $\sum_{i+j \geq 2} b_{ij} x^i y^j$ converges for all sufficiently small $|x|$ and $|y|$. The boundary condition is $u(t) = g(t)$ for $t_0 \leq t \leq t_0 + 1$, where $g(t)$ is a given continuous function. Bellman (Ann. of Math. vol. 50 (1949) pp. 347–355) has studied (2) in case $a(t) \to a_0$ and $b(t) \to b_0$ as $t \to +\infty$. Using transform methods, the author shows that if $a(t) \sim a_0 + (a_1/t) + \cdots$ and $b(t) \sim b_0 + (b_1/t) + \cdots$ as $t \to +\infty$, the linear equation (1), subject to the stated boundary condition, has a unique solution for $t > t_0$. If S denotes the root of largest real part of the equation (3) $se^s - b_0 e^s - a_0 = 0$, and if δ is the residue of $(a_1 e^{-s} + b_1)/(s - b_0 - a_0 e^{-s})$ at S, this solution is $O(t^{\mathrm{Re}(\delta)} e^{\mathrm{Re}(S)t})$ as $t \to +\infty$. If all the roots of (3) have negative real parts and if $|g(t)|$ is sufficiently small over the initial interval, the same conclusions are valid for (2). (Received April 29, 1952.)

533. H. A. Dye: *The unitary structure in finite rings of operators.*

A W^*-algebra (= weakly-closed self-adjoint algebra of operators on Hilbert space) is called *non-atomic* if it contains no minimal projections. By means of a theorem of Liapounov on the range of non-atomic vector-valued measures, it is shown that (i) the weak closure of the set of projections in any non-atomic W^*-algebra fills out the positive part of the unit sphere, and (ii) the weak closure of the set of unitary operators in a non-atomic W^*-algebra containing no purely infinite projections fills out the entire unit sphere. This fact is then used to characterize the algebraic and spatial types of a finite non-atomic W^*-algebra M in terms of its unitary structure. Here, by definition, the *unitary structure* in M is the group M_U of unitary operators in M viewed as a uniform space in the uniform structure induced by the weak topology. Two such structures M_U and N_U are called isomorphic if there exists a biunivalent mapping ϕ of M_U on N_U which is a group isomorphism, a unimorphism, and which satisfies $\phi(iI) = iI$, where $i = (-1)^{1/2}$. Each *-isomorphism between two non-atomic finite W^*-algebras M and N induces an isomorphism between their unitary structures, and conversely, an isomorphism between the unitary structures in M and N has a unique extension to a *-isomorphism of M on N. A corresponding notion of strong isomorphism is defined between unitary structures which serves to characterize the corresponding rings up to unitary equivalence. (Received May 2, 1952.)

534*t*. Abolghassem Ghaffari: *On the existence of limit cycles of a certain nonlinear differential equation.*

The author considers the nonlinear differential equation (1) $dy/dx = [y(x^2+y^2-2x-3)(x^2+y^2-2x-8)+x][x(x^2+y^2-2x-3)(x^2+y^2-2x-8)-y]^{-1}$ and discusses the behavior of its characteristics in the large. It is shown that the only critical point of (1) is the *origin* which is an *unstable focus*. In polar coordinates ρ, θ, the equation (1) transforms into (2) $d\rho/d\theta = \rho(\rho^2-2\rho\cos\theta-3)(\rho^2-2\rho\cos\theta-8)$, which is discussed incompletely by H. Poincaré (*Oeuvres*, vol. I, p. 83). Using as Poincaré's topographical system the circles $\rho=$constant, one finds that the curves of contacts are the origin and the circles $\rho^2-2\rho\cos\theta-3=0$ and $\rho^2-2\rho\cos\theta-8=0$, so that the regions $0<\rho<1$, $\rho>4$ are *acyclic* and the annular region $1<\rho<4$ is doubtful. As the circles $\rho_1=1$, $\rho_4=4$ are of the same sign (positive), there exists an even number of limit cycles in the region $1<\rho<4$. The doubtful region can be divided into two smaller doubtful regions $\rho_1<\rho<C$ and $C<\rho<\rho_4$ by the auxiliary circle without contacts C: $\rho^2-2\rho\cos\theta-11/2=0$. It is found that the sign of C is negative and thus different from those of ρ_1 and ρ_4; therefore each of the two smaller doubtful regions is a *monocyclic* region. Then the equation (1) has, besides the equator, two limit cycles and the characteristics are composed of three families of curves spiralling away from the origin and the limit cycles. (Received May 2, 1952.)

535. I. M. Hostetter: *Boundary value solutions from the general solution of certain partial differential equations.* Preliminary report.

Let R be the position vector of a point R of space and $f(M\cdot R)$ an arbitrary function. Direct substitution of $u=f(M\cdot R)$ in the homogeneous space linear differential equation $\Phi:\nabla\nabla u=0$ shows that for f to be a solution M must satisfy the condition $M\cdot\Phi\cdot M=0$. If Φ is nonsingular and definite then M must be of the form $P+iQ$ where P and Q satisfy the conditions $P\cdot\Phi\cdot P=Q\cdot\Phi\cdot Q$, $P\cdot\Phi\cdot Q=0$. The totality of all solutions $f(P\cdot R+iQ\cdot R)$ where P and Q satisfy these conditions is the general solution of $\Phi:\nabla\nabla u=0$. If u is defined on a boundary σ, let Q, or P, depending upon the nature of the solution sought, represent the points of the boundary. If, say, Q_j are a set of n such points, then $u=a\sum_{j=1}^{n} f(P_j\cdot R+iQ_j\cdot R)\Delta\sigma$, and consequently $u=a\int_\sigma f(P\cdot R+iQ\cdot R)d\sigma$, is a solution. If f is properly chosen, the boundary conditions will be satisfied. The method is extended to obtain the solution of boundary value problems of special types of nonhomogeneous equations and leads directly and simply to the usual solutions of the standard boundary value problems. (Received May 8, 1952.)

536*t*. M. S. Klamkin: *Generalization of Clairaut's differential equation and the analogous difference equation.*

Clairaut's equation is generalized to $F(z_0, z_1, \cdots, z_{n-1})=0$ where $z_r = \sum_{s=0}^{n-r-1} (-1)^s x^s y^{(r+s)}/s!$, $y^{(r)}=d^ry/dx^r$. The solution is obtained by differentiating $F=0$, which yields $d^ny/dx^n=0$. Thus $y=\sum_{r=0}^{n-1} a_r x^r/r!$ where $F(a_0, a_1, \cdots, a_{n-1})=0$. The analogue of the generalized differential equation as a difference equation is $F(z_0, z_1, \cdots, z_{n-1})=0$ where $z_r=\sum_{s=0}^{n-r-1}(-1)^s C_{x+s-1,s}\,\Delta^{r+s}u_x$. The solution is obtained by considering $\Delta F=0$, which yields $\Delta^r u_x=0$. Thus $u_x=\sum_{r=0}^{n-1} a_r C_{x,r}$ where $F(a_0, a_1, \cdots, a_{n-1})=0$. (Received May 7, 1952.)

537*t*. V. L. Klee: *Convex sets in linear spaces.* III

(The first two papers of this series appeared in Duke Math. J. vol. 18 (1951) pp. 443–466 and 875–883.) Let L be a linear system and T the collection of all sets $X \subset L$ such that for each finite-dimensional linear variety $V \subset L$, $X \cap V$ is open in the natural "Euclidean" topology of V. It is proved that the linear space (L, T) (L topologized by T as its family of open sets) is locally convex if and only if $\dim L \leqq \aleph_0$; and for $\dim L > \aleph_0$, L actually contains a closed convex proper subset which intersects every nonempty open convex subset. This answers negatively a question raised earlier [loc. cit., p. 447, (Q_2)]. Supplementing an earlier result [loc. cit., p. 462, (12.4)], the following are proved: $\dim L < \aleph_0$ if and only if every p^+-functional on L majorizes at least one linear functional; $\dim L \leqq \aleph_0$ if and only if every lower semi-continuous p^+-functional on L majorizes at least one linear functional. (Received May 8, 1952.)

538. R. B. Leipnik: *New existence proof for semi-linear elliptic equations.*

Axiomatic inversion theory (cf. *Axiomatic Perron inversion*, Proceedings of the International Congress of Mathematicians, Cambridge, 1950, Vol. I, p. 462) is applied to semi-linear elliptic partial derivative equations to obtain existence theorems. The results are in some respects stronger than those based on the theory of completely continuous operators in Banach space. (Received May 7, 1952.)

539t. Norman Levinson: *The treatment of the irregular singular point for differential equations by the Phragmén-Lindelöf theorems.*

The system of differential equations $dw/dz = z^r A(z)\, w$, where w is a vector, z a complex variable, r an integer $\geqq 0$, and $A(z) = A_0 + A_1 z^{-1} + A_2 z^{-2} + \cdots$ with A_0, $A_1, \cdots$ constant matrices, can have $z = \infty$ as an irregular singular point. Formal solutions are known to exist. Recasting the system into a system of integral equations, by a well known procedure, the existence along any radial direction (or similar curve) of actual solutions for which the formal solutions are asymptotic is well known. The proof that the actual solutions are represented asymptotically by the formal solutions in certain sectors of the complex plane, due in the general case to Trijitzinsky, is more complicated. It can be shown that by use of familiar Phragmén-Lindelöf theorems (Problems 331 and 340 of Vol. 1, part III, of *Aufgaben und Lehrsätze* by Pólya and Szegö) the result along radial directions (or similar curves) implies the result in the appropriate sectors of the complex plane, thereby simplifying the proof considerably. (Received March 13, 1952.)

540. A. E. Livingston: *Some Hausdorff means which exhibit the Gibbs' phenomenon.*

The regular Hausdorff means of order n, with kernel $g(x)$, for the sequence $d_n(x) = \sum_{k=1}^{n} \sin kx/k$ are given by $h_{n,g}(x) = \sum_{k=1}^{n} C_{n,k} d_k(x) \int_0^1 t^k (1-t)^{n-k} dg(t)$, with $g(0+) = g(0)$, $g(1) - g(0) = 1$, and $g(x)$ of bounded variation on the interval $0 \leqq x \leqq 1$. Otto Szász (Trans. Amer. Math. Soc. vol. 69 (1950) pp. 440–456) has shown that if $x_n \to 0+$ and $nx_n \to a \leqq \infty$ as $n \to \infty$, then $h_{n,g}(x_n) \to \int_0^1 \mathrm{Si}\,(ax) dg(x)$ ($\mathrm{Si}\,(x) = \int_0^x (\sin t/t dt)$. Define $F(g) = \max_{a>0}\, (2/\pi) \int_0^1 \mathrm{Si}\,(ax) dg(x)$. If $F(g) > 1$, then the sequence $h_{n,g}(x)$ exhibits the Gibbs' phenomenon. The author proves that if the N points of jump of the step-function kernel $s_N(x)$ are linearly independent over the rationals, then $F(s_N) > 1$. For $N = 2$, he shows that every $s_2(x)$ satisfies $F(s_2) > 1$. (Received February 18, 1952.)

541*t*. E. R. Lorch: *A curvature study of convex bodies in Banach spaces.*

Let $\mathfrak{B}$ be a real Banach space with norm $\|x\|$. Let $r>1$. Write $G(x)=r^{-1}\|x\|^r$. Suppose that $\phi(\alpha, \beta)=G(x+\alpha y+\beta z)$ is twice continuously differentiable. Write the derivatives at $\alpha=\beta=0$ as $G_y(x)$, $G_{yz}(x)$, etc. Suppose that for $x\neq 0$, $y\neq 0$, $G_{yy}(x)>0$ (implying that the curvature of the unit sphere in $\mathfrak{B}$ is positive). It is shown that $G_y(x)$ is a bounded linear functional $f(y)$ in $\mathfrak{B}^*$ with $\|G_y(x)\|=\|x\|^{r-1}$. Further, $G_{yz}(x)$ is bilinear in y, z and is a bounded linear functional $g(z)$ for each y. If $x\neq 0$ is fixed, the mapping $y\rightarrow G_{yz}(x)$ is 1-1 bounded linear of $\mathfrak{B}$ into $\mathfrak{B}^*$. If $f(y)\in\mathfrak{B}^*$ assumes its l.u.b. at x, then $f(y)=\lambda G_y(x)$, $\lambda>0$. If there is an $L>0$ such that $L\|x\|^{r-2}\|y\|^2\leqq G_{yy}(x)$, then $\mathfrak{B}$ is uniformly convex and the mapping $G_y(x)\rightarrow x$ is 1-1 strongly continuous (nonlinear) from $\mathfrak{B}^*$ onto $\mathfrak{B}$. Also the mapping $y\rightarrow G_{yz}(x)$ for fixed $x\neq 0$ is bicontinuous linear of $\mathfrak{B}$ onto $\mathfrak{B}^*$. If for $L>0$, $K>0$, $L\|x\|^{r-2}\|y\|^2\leqq G_{yy}(x)\leqq K\|x\|^{r-2}\|y\|^2$ (implying curvature of unit sphere is bounded away from 0, ∞), $\mathfrak{B}$ is isomorphic to a Hilbert space. The mapping $x\rightarrow G_y(x)$ is a strong homeomorphism of $\mathfrak{B}$ onto $\mathfrak{B}^*$ which maps the unit sphere of $\mathfrak{B}$ onto that of $\mathfrak{B}^*$. (Received May 6, 1952.)

542. O. M. Nikodým: *On accessibility in convex sets in abstract linear spaces.*

Let E be a convex set containing at least two points, in abstract linear infinite-dimensional real space L where no topology is supposed. By lin E we understand, with V. L. Klee, Jr., the set of all points $y\in L$ such that there exists $x\in E$ where the half-open rectilinear segment $[x, y)$ belongs to E. Klee has given an example of E where $\text{lin}^2 E\neq\text{lin } E$. The question whether the relation $\text{lin}^3 E\neq\text{lin}^2 E$ is possible has not been answered yet. In this respect the following theorems hold true: I. For every ordinal α where $I\leqq\alpha<\Omega$ there exists E with $\text{lin}^\alpha E\neq\text{lin}^{\alpha+1} E$. II. If the Hamel basis of L is countable, then for every E there exists $\alpha<\Omega$ with $\text{lin}^\alpha E=\text{lin}^{\alpha+1} E$. III. If the Hamel basis is not countable, there exists E with $\text{lin}^\alpha E\neq\text{lin}^{\alpha+1} E$ for every $\alpha<\Omega$. IV. For every E we have $\text{lin}^\Omega E=\text{lin}^{\Omega+1} E$. V. The linear closure of E coincides with $\text{lin}^\Omega E$ and is a convex set. This paper is part of the work under a cooperative contract between A.E.C. and Kenyon College. (Received May 5, 1952.)

543*t*. H. S. Shapiro: *Maximum-minimum duality in extremal problems.* Preliminary report.

It is known that certain maximum problems in the theory of functions can be associated with minimum problems, and solved by exploiting this interrelation. For example, Macintyre and Rogosinski, Acta Math. vol. 82, state the result: "Let $K(z)$ be regular in $|z|<1$ except for a finite number of poles, and $\lim\sup_{r\rightarrow 1-0}\|K(re^{i\theta})\|_{p'}<\infty$. Let $I(f)=|(1/2\pi i)\int_{|z|=1}f(z)K(z)dz|$ and $J(g)=\|K+g\|_{p'}$. Then max $I(f)$ over all $f\in H_p$ with $\|f\|_p\leqq 1$ equals min $J(g)$ over all $g\in H_{p'}$. The max and min are attained by f_1 and g_1 such that (1) $|(1/2\pi i)\int_0^{2\pi}f_1(e^{i\theta})K_1(e^{i\theta})d\theta|=\|f_1(e^{i\theta})\|_p\|K_1(e^{i\theta})\|_{p'}$, where $K_1=K+g_1$." From (1) much further information is then deduced, leading in many special cases to the determination of f_1, K_1, and max $I(f)$. (In the above, $\|F\|_p=\|F(\theta)\|_p$ denotes $((1/2\pi)\int_0^{2\pi}|F(\theta)|^p d\theta)^{1/p}$; H_p is the class of $f(z)$ regular in $|z|<1$ for which $\lim\sup_{r\rightarrow 1-0}\|f(re^{i\theta})\|_p<\infty$; $1\leqq p\leqq\infty$, and $1/p+1/p'=1$.) The author gives a simple proof of the above theorem, using the Hahn-Banach theorem and F. Riesz' theorems on the representation of linear functionals over the space L_p. The generality of the author's method permits him to extend the above result in several ways, for example: more general kernels K, more general domains than the unit circle, and replacement of the class H_p by a corresponding class of polynomials. The underlying reason for maximum-minimum duality is seen to be the fact that the norm of a linear functional relative to a subspace $\mathfrak{X}_1$ of a Banach space $\mathfrak{X}$ equals the distance in the conjugate space $\mathfrak{X}^*$ of T from the annihilator of $\mathfrak{X}_1$. (Received May 21, 1952.)

544. Andrew Sobczyk: *Generalized measures and functionals of testing functions.*

A continuous linear functional $f(\phi)$ on the space (m) of bounded sequences is of the form $f(\phi) = \int \phi(t) dF$, where F is a finitely additive set function on all subsets of the set T of positive integers. If $\phi_X(t)$ is the characteristic function of $X \subset T$, then $F(X) = f(\phi_X)$. For a nonlinear functional $g(\phi)$ on (m), the equation $G_a(X) = g(a\phi_X)$ associates with $g(\phi)$ a one-parameter family of set functions; in this paper set functions are studied to advantage from the point of view of functionals on (m). Characterizations are obtained of convex, quasi-convex, monotone, subadditive, finitely and countably-many valued set functions, in terms of the associated functionals. Properties are determined of the class of two-valued set functions modulo the class of two-valued additive set functions. Similarly nonlinear functionals on the linear topological space $(\mathfrak{D})$ of the Schwartzian theory of distributions, and how they correspond to nonadditive set functions and multiple-layers, are considered. Other spaces of testing functions than $(\mathfrak{D})$ and (m), appropriate for other families of sets than Borel fields and the family of all subsets of T, also are introduced, and the corresponding functionals and generalized measures are studied. (Received May 7, 1952.)

545. G. E. Uhrich: *A generalized hyperbolic differential equation.* Preliminary report.

An examination is made of the partial differential equation $D_0 u = f(x_1, \cdots, x_n; u, D_1 u, \cdots, D_m u)$ in which $D_p u$ $(p = 1, \cdots, m)$ is a partial derivative of the dependent variable u with respect to the independent variables $x_1, \cdots, x_n$, the order of $D_0 u$ exceeds that of each of the other derivatives involved, and further its order with respect to x_k $(k = 1, \cdots, n)$ is not exceeded by the order with respect to x_k of any of the other derivatives. When a certain set of functions called "initial determinations" are given, and when these functions together with the function f of the equation satisfy a set of specified conditions, the existence of a unique solution $u = u(x_1, \cdots, x_n)$ of the differential equation is insured. (Received May 8, 1952.)

546. J. L. Ullman: *A notion of capacity for distributions.*

In an investigation of the relationship of Hankel determinants whose elements are the coefficients of a Taylor series to the singularities of the corresponding function, it was shown that the following notion of average capacity played a significant role. Let $\mu(t)$ be a nondecreasing function, constant outside the interval (0, 1), let E be the set of t for which $\mu(t+\epsilon) - \mu(t-\epsilon) > 0$ for all $\epsilon > 0$, and let E_ρ be the subset of E in the closed interval $(\rho, 1)$. The average capacity of E_ρ, denoted by $\bar{\tau}(E_\rho, \mu)$, is defined as $\limsup_{k\to\infty} |\overline{M}_k|^{i/k(k-1)}$, where $\overline{M}_k = \int_\rho^1 \cdots \int_\rho^1 V(t_1, \cdots, t_k) d\mu(t_1) \cdots d\mu(t_k)$, $V(t_1, \cdots, t_k) = \prod_1^k \prod_1^k {}_{i \neq j} (t_i - t_j)$. Let λ be the lower bound of ρ for which $\bar{\tau}(E_\rho, \mu) = 0$, and let $f(z) = \int_0^1 \mu(t)/(t-z)$. The problem is to give a function-theoretic characterization of the behavior of $f(z)$ in a neighborhood of E_λ. The following represents a partial solution, namely, it is shown that $\lim_{y\to 0} f(x+iy) - f(x-iy) = 0$ for almost all x in the interval $(\lambda, 1)$. This is proved by first showing that $\mu(t)$ has derivative zero except for a set of measure zero in the interval $(\lambda, 1)$, and an important step is the generalization of a relationship due to Edrei (Compositio Math. (1939), Proposition 7, p. 65). (Received May 9, 1952.)

547t. Albert Wilansky: *The basis in summability space.*

Let Φ be the set $\{i, \delta^1, \delta^2, \delta^3, \cdots\}$, where $i=\{1, 1, 1, \cdots\}$, $\delta^k=\{0, 0, \cdots, 0, 1, 0, \cdots\}$ (1 in the kth place). S. Mazur (Studia Mathematica vol. 2) proved that Φ is fundamental in the field of a summability matrix if and only if it is of type M. A matrix A is said to have the PMI property if for every equipotent B, $\sum b_k x_k$ converges for every x summable-B, where $b_k=\lim_n a_{nk}$. The $(C, 1)$ matrix has the PMI property, the Nörlund matrix ($p_0=p_1=1$, $p_k=0$ for $k>1$) has not. Theorem. *Φ is a basis for the field of A if and only if A has the* PMI *property.* Principal tool: a known lemma on biorthogonal sets and the result: A has the PMI property if and only if $|\sum_{k=1}^m a_{nk}x_k|<M$, M depending on x, for all m, n and each x which is summable A. (Received April 23, 1952.)

Applied Mathematics

548. Gertrude Blanch: *Note on the numerical solution of differential equations involving linear differential operators.*

This note is concerned with the numerical solution of equations of the type $Ty+F(y, x)=0$, where T is a linear differential operator, and initial conditions are given at the origin of the system. It is shown that when the solutions of the homogeneous equation are known, then the computations can be so arranged that the contribution to the integral from the upper limit, at each step of the numerical process, when y is still unknown at the upper limit, drops out. The method involves, essentially, using the integral representation of the system. Thus it is not necessary to employ the usual method of first approximating y by an open integration formula and then iterating the solution until the desired accuracy is obtained; the solution can be obtained by mere quadratures. The method is especially successful in the important case when the differential operator has constant coefficients and is of low order. An example is given, and it is shown that for some range of parameters entering into the differential operator, the method of using the integral representation leads to numerical processes that are easier to manage than the ordinary routine for stepwise integration. (Received May 5, 1952.)

549. R. G. Selfridge: *Approximations with least maximum error.*

For certain classes of approximating functions, where each class forms an n parameter family, it has been shown that the approximation with the least maximum error to a function on a given interval is that with its error curve having absolute extrema of alternating sign, and with at least $n+1$ such absolute extrema. This paper shows that this is also a sufficient condition for least maximum error for a different class of approximating functions. An iteration process is described that has been used to find the approximations, but no proof of convergence is given. (Received April 24, 1952.)

550. W. M. Stone: *A form of Newton's method with cubic convergence.*

Using the generalized Taylor expansion of Hummel and Seebeck (Amer. Math.

Monthly vol. 56 (1949) pp. 243–247), a formula is presented for the root of an equation, $f(x)=0$, in terms of values of $f(x)$ and $f'(x)$ at points on either side of the root. Advantage lies in the property of cubic convergence without the necessity of evaluating second derivatives; if $f(x)$ is a scantily tabulated transcendental function, considerable interpolation may be avoided. (Received May 19, 1952.)

GEOMETRY

551t. C. M. Fulton: *The projecting tensor.*

This paper deals with a Riemannian space V_n of n dimensions imbedded in a Riemannian space V_m of m dimensions. The projecting tensor is defined as a tensor belonging to the subspace V_n. Given any tensor at a point of V_n, its projection on V_n can be found with the aid of the projecting tensor. Proper multiplication and subsequent contraction of these tensors yield the projection. The procedure also leads to a simple interpretation of the covariant derivative. (Received April 16, 1952.)

552t. P. C. Hammer: *Diameters of convex bodies.* Preliminary report.

A convex body C in E_n is taken to be a closed bounded convex set with interior points. A diameter of C may be considered as a chord of C which is a longest chord in its direction. The ratio in which an interior point of C divides a chord through it is taken as the ratio of the larger (or equal) part to the whole. The maximum such ratio for a point x is designated by $r(x)$. Two results are: (1) If x divides a chord ab of C in the ratio $ax/ab=r(x)$, then if y is any point on xb, $ay/ab=r(y)$. (2) Every diameter of a convex body C is intersected by c other diameters of C. Since it is known that the chord ab mentioned above is necessarily a diameter, if we designate by $F(r)$ the set of diameters of C each of which is divided by a point in its maximum ratio $r<1$ then the family of sets $F(r)$ descends as r decreases to the minimum value, r_0, of $r(x)$. (Received April 18, 1952.)

553t. P. C. Hammer and Andrew Sobczyk: *Planar line families.* III.

A family F of lines is said to be outwardly k-fold if it covers points exterior to some circle continuously precisely k-fold, including points at infinity. An outwardly k-fold family may be represented by the equations (1) $x \sin \alpha - y \cos \alpha - p_i(\alpha)=0$, where $i=\pm 1, \pm 2, \pm n$ if $k=2n$, or $i=0, \pm 1, \pm 2, \cdots, \pm n$ if $k=2n+1$. Necessary and sufficient conditions that equations (1) represent an outwardly k-fold line family are: (a) $p_i(\alpha)$ are continuous for $0 \leqq \alpha \leqq \pi$, (b) the upper and lower right and left derivatives of $p_i(\alpha)$ are uniformly bounded, (c) $p_i(0)=-p_{-i}(\pi)$, $i=\pm 1, \pm 2, \cdots, \pm n$, and also $i=0$ if k is odd, (d) it is possible to choose the sign of subscripts so that $p_i(\alpha)>p_j(\alpha)$ for $j<i$ and for all α in $[0, \pi]$. (Received April 18, 1952.)

TOPOLOGY

554t. D. O. Ellis and H. D. Sprinkle: *Topology in B-metrized spaces.* Preliminary report.

Let B be a σ-complete Boolean algebra. A set S with a distance function taking values in B and having the formal properties of a metric is called a *B-metrized space.* An example is B itself with the autometrized distance function (David Ellis, *Autom-*

etrized Boolean algebras I, Canadian Journal of Mathematics vol. 3 (1951) pp. 87–93). The distance topology induced in a B-metrized space (see David Ellis, *Geometry in abstract distance spaces*, Pub. Math. Debrecen vol. 2 (1951) pp. 1–25) by the Kantorovitch topology of B is considered. Cauchy sequences, completeness, and related questions are studied. As an interesting application the following are demonstrated: A topological space is 0-dimensional if and only if it is a space of uniform structure defined by a filter having a symmetric idempotent base. Any T_1 space, S, whose topology is compatible with such a uniform structure is metrizable over the Boolean algebra $2^{S \times S}$. One also finds that B itself under its autometrized topology is always complete (this was shown by Löwig using a different definition of Cauchy sequence than ours). (Received April 17, 1952.)

555*t*. S. T. Hu: The homotopy addition theorem.

The homotopy addition theorem is a fundamental assertion which relates the element of the homotopy group determined by a map of a cellular sphere to those determined by the partial maps on the cells of the sphere. It is used in the proof of the Hurewicz Isomorphism Theorem and is also a consequence of it. However, in the mathematical literature, one can not find an explicit proof of this elementary but important assertion concerning homotopy groups. In the present paper, an elementary proof of the homotopy addition theorem is given. It is elementary in the sense that it is based only on the definition of homotopy groups together with a few well known properties of homtopy which were elementarily proved in the mathematical literature. The proof is given in a completely detailed and formalized way. (Received May 2, 1952.)

556. Tibor Radó and P. V. Reichelderfer: *Remarks on the topology of Euclidean spaces.*

The purpose of this note is to indicate simple methods by which many of the classical theorems in the topology of Euclidean n-space may be derived from the eight basic axioms for the relative cohomology groups of compact pairs of Hausdorff spaces proposed by S. Eilenberg and N. E. Steenrod in *Foundations of algebraic topology*, Princeton University Press. First, an elementary proof is given for the theorem that the cohomology group $H^n(F)$ of a compact subset F of Euclidean n-space is trivial. Almost as a corollary it is shown that if F is a compact subset of Euclidean n-space for $n \geqq 2$ such that the complement of F is connected, then $H^{n-1}(F)$ is trivial. This result is used to derive a proof of the invariance of the interior and of the frontier in Euclidean n-space. (Received April 21, 1952.)

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