BOOK REVIEWS

Linear algebra and projective geometry. By R. Baer. New York, Academic Press, 1952. 8+318 pp. $6.50.

A half-century or so has elapsed since the great treatises on foundations of geometry appeared. In these works it was taken for granted that the reader could stand any amount of intricate geometrical reasoning; but the idea that geometry could be done over an arbitrary division ring was quite a novelty and was accordingly treated with respect and caution. In the present book we see how much the mathematical climate has changed. The necessity of grappling with an arbitrary division ring should be the least of the reader's worries. Nor will a lack of geometric intuition seriously impede him (it is interesting to note that there are only nineteen figures in the book, of which two are non-geometrical and the last seven are concerned with the introduction of coordinates). But a good backlog of experience with the trickery of modern algebra is recommended to any prospective reader.

Let F be a division ring, A a vector space over F, both more or less arbitrary. While it is true that the case of characteristic two often gets "cavalier treatment," possible non-commutativity of F is allowed every scope for its operations. Infinite dimensionality of A is permitted, but plays a subdued role, coming to the reader's attention mainly when considerations of duality make it impossible. With a minimum of delay the object of central interest makes its appearance: the lattice L of subspaces of A. If L_1 is a second such lattice, call a lattice isomorphism between L and L_1 a projectivity. The first fundamental theorem asserts that any projectivity is induced by a semilinear transformation, provided the dimension of A is at least three.

A lattice isomorphism of L upon itself is an auto-projectivity. In the group of auto-projectivities we can pick out the subgroup of collineations, consisting of those auto-projectivities which can be induced by linear transformations. A pitfall awaits us here, for the same projectivity may be induced by both a linear and a semi-linear transformation, provided the automorphism for the latter is inner. This is the first of many occasions on which inner automorphisms have to be accorded special consideration. It is incidentally gratifying that perspectivities, which play such an inflated part in classical accounts, are here cut down to their proper role of being possible building blocks of collineations. The discussion moves on to the second fundamental theorem (fixing a collineation by its effect on a simplex), Pappus's theorem, and cross ratio.

A lattice anti-isomorphism (order-inverting map) of L upon L_1 is a duality, and a duality of L upon itself is an auto-duality. Chapter IV (the longest in the book) is devoted to their study. The theory of auto-dualities turns out to be almost identical with that of semi-bilinear forms; we say "almost" because the forms $f(x, y)$ and $f(x, y)c$ induce the same dualities. Of major interest are polarities, which are auto-dualities of period two, the corresponding algebraic gadgets being alternate and Hermitian forms. The structure of these forms and of groups over the forms is discussed in considerable detail.

In Chapter V the ring of linear transformations is taken up. This is perhaps somewhat less novel than the remaining chapters, for in the last decade algebraists have been operating successfully in somewhat more general rings. Next the groups of non-singular linear and semi-linear transformations are studied, and their isomorphisms completely determined. The method, originated by Mackey, is to use the elements of order two to reconstruct the vector space.

The final chapter introduces coordinates in a projective geometry, assumed to be Desarguesian if it is a plane. The method is to prove directly that the given geometry can be embedded as a hyperplane H in a larger geometry K, a tour de force which may leave the reader slightly stunned. Then the desired vector space is obtained by manipulating the group of collineations of K leaving H pointwise fixed, an idea stemming from Dehn and Artin. Happily, the author concludes with some indications of how one can detour around the embedding theorem.

The author has done much more than collecting and proving elegantly theorems already in the literature. Here is just a sample of important results difficult or impossible to locate elsewhere. (1) If A is a c-dimensional vector space over a division ring with d elements, then the dimension of the full dual of A is d^c. (2) The collineation group is generated by the projectivities which have a line of fixed points. (3) Even without commutativity, cross ratio is definable up to an inner automorphism. The formula is

$$(h-n)(k-n)^{-1}(k-m)(h-m)^{-1}.$$

(4) The generalized theorem of von Staudt-Ancochea: if a mapping on a line preserves one central cross ratio, other than 0 or 1, it is induced by an automorphism or anti-automorphism. (5) An auto-duality is of period two if and only if the corresponding semi-bilinear form has the property that orthogonality is symmetric; and the form may then be normalized to be alternate or Hermitian. (6) Sylvester's theorem of inertia, generalized to any division ring with

suitable order properties. (7) Pascal's theorem holds only for the usual symmetric forms and in particular requires commutativity (conic sections make a fleeting appearance at this point in the book). (8) Determination of the two-sided ideals in the ring of all linear transformations, there being one for each cardinal number.

There are two generalizations which the author calls to the reader's attention. The first is to replace the vector space by a suitable kind of module over a ring; in a series of papers the author himself has carried this program nearly to completion. The second is to replace the vector space (implicity paired to its full dual) by an arbitrary pair of dual vector spaces; here there has been substantial work by Mackey and Rickart. Further in the distance lies the project of uniting these two generalizations by studying dual modules. In yet a different direction lie the still largely mysterious rings and lattices without minimal elements, typified by von Neumann's continuous geometry. So there is much to be done; and the coming generations of young algebraists, with this book happily tucked under their arms, will find the path well laid out.

I. KAPLANSKY

Calculus of variations with applications to physics and engineering. By Robert Weinstock. New York, McGraw-Hill, 1952. 10+326 pp. $6.50.

This book, which appears in the International Series in Pure and Applied Mathematics, has been written to fill the need for an elementary introduction to the calculus of variations, followed by extensive applications to physics and theoretical engineering. By far the greater emphasis is placed on the applications, and the list of chapter headings will show the scope: 1. Introduction; 2. Background preliminaries; 3. Introductory problems; 4. Isoperimetric problems; 5. Geometrical optics: Fermat's principle; 6. Dynamics of particles; 7. Two independent variables: the vibrating string; 8. The Sturm-Liouville eigenvalue-eigenfunction problem; 9. Several independent variables: the vibrating membrane; 10. Theory of elasticity; 11. Quantum mechanics; 12. Electrostatics.

A book with this scope should have a wide appeal at the present time, particularly among those physicists and engineers who find variational methods tricky and evasive. For the author's aim is clarity of exposition. He goes slowly at the beginning, where slowness is essential, and he provides, at the ends of the chapters, sets of exercises which should prove very useful. He is writing for those who know the concepts and techniques of a first year calculus course,