BOOK REVIEWS

Sur quelques propriétés des valeurs caractéristiques des matrices carrées. By M. Parodi. (Mémorial des Sciences Mathématiques, no. 118.) Paris, Gauthier-Villars, 1952. 64 pp. 800 fr.

This book deals in the first place with the applications of the following determinant theorem: Let $A=(a_{ik})$ be a square matrix of order n with real or complex elements and let $|a_{ii}| > \sum_{k\neq i}|a_{ik}|$ $(i=1, \cdots, n)$. It follows that the determinant $D=|a_{ik}|\neq 0$. This theorem has varied applications and is being rediscovered continually. Parodi mentions some of its discoverers and some of its generalizations without making any attempt at completeness. In particular, Price's recent improvement of Ostrowski's lower bound for $|D|$ is not mentioned.

A most important application of this determinant theorem lies in its usefulness for the bounding of characteristic roots of matrices. For, it implies that the characteristic roots of A lie inside or on the boundaries of the n circles $|a_{ii}-z| = \sum_{k\neq i}|a_{ik}|$. A number of applications and generalizations are treated, in particular that due to A. Brauer in which the n circles are replaced by $n(n-1)/2$ Cassini ovals. The references are again incomplete; in particular, the pioneering work of Gershgorin is not mentioned.

Parodi exploits a theorem of Ostrowski which deals with the variations of the elements which preserve the nonsingularity of a matrix.

The book concludes with the study of the determinantal equation of the form

$$\left| a_{ik}^{(0)} z^n + a_{ik}^{(1)} z^{n-1} + \cdots + a_{ik}^{(n)} \right| = 0$$

where $(a_{ik}^{(j)})$ are square matrices; the case $n=2$ which turns up in electrical networks is treated earlier and upper bounds for the real parts of the roots z are determined.

Apart from various applications to circuit theory, the book deals with general stability problems, with the reducibility of polynomials, and with methods of finding bounds for the modules of the zeros of polynomials. It contains a number of worked numerical examples of matrices of small orders.

O. Taussky

Complex analysis. By L. V. Ahlfors. New York, McGraw-Hill, 1953. 12+247 pp. $5.00.

In American universities the course in complex analysis is often used to review advanced calculus with modern standards of rigor. This procedure, which clearly has certain disadvantages, is to some extent forced on our graduate schools, and it is for such a course that this text is written. In complex analysis the temptation to base proofs on intuitive arguments is especially strong. It is a challenge to make the proofs simple and intuitively clear, and at the same time such that the reader can easily fill in the steps of a formal argument. The author has met this challenge in a masterful way.

Roughly the first third of the book is concerned with notions preliminary to complex integration. In it complex numbers are defined by means of an extension field, after which they are plotted in the plane, which is in turn projected onto a sphere. The elementary functions log x, arc sin x, etc., are defined for real values of the variable by means of appropriate integrals, from which the properties of these functions and their inverses are derived. These functions can then be defined for complex values of the variable in terms of their real and imaginary parts. The complex derivative is defined and the notion of conformality introduced. By this time the reader has studied several mappings in the large, including the set of bilinear functions, and has some notion of the local properties of analytic functions. This part of the book ends with an intuitive discussion of Riemann surfaces which, as the author states, is only for purposes of illustration.

The treatment of complex integration, which has been influenced by the work of Artin, is novel in several respects. If the continuous function $z(t)$ has a piecewise continuous derivative in the interval $a \leqq t \leqq b$, then $z(t)$ describes an arc γ when t describes $[a, b]$. If $f(z)$ is continuous on γ, then the integral on $f(z)$ over γ is defined in terms of a real integral by

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

In this definition it is not assumed that γ is a Jordan arc nor that $z'(t)$ is different from zero. It is then shown that the line integral $\int_\gamma(pdx+qdy)$ defined in a region (open connected set) depends only on the end points of γ if and only if there is a function $U(x, y)$ such that $p=U_x$, $q=U_y$. In particular the integral of the functions 1, z around a rectangle is zero. This result is then used to prove Cauchy's theorem for a rectangle by the method of dissection. Cauchy's theorem for any closed curve γ (with the same smoothness conditions as above) in an open disk is a consequence of these results. For it can now be shown that if $f(z)$ is regular in an open disk, then the function

$$F(z) = \int_\sigma f(z)dz$$

is well defined in the disk. Here σ is a path from the center of the disk to a point z, consisting of one vertical and one horizontal line segment. There are two such paths corresponding to a single point z, and their difference is a rectangle.

In the case of Cauchy's integral formula some notion of the topology of the path of integration must be present, implicitly or explicitly. In order to separate the topology from the analysis so far as possible the author makes use of the winding number. If γ is a closed curve and a is a point not on γ, then the winding number $n(\gamma, a)$ is what the name implies, the number of times that γ winds around the point a. If γ is a closed curve lying in a disk in which $f(z)$ is analytic, then Cauchy's integral formula is

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz.$$

Criteria can be given that $n(\gamma, a)$ is equal to 1 (or zero) which are easier to prove, although less general, than the Jordan curve theorem.

This information is then used to investigate a variety of topics, which include Taylor's series, the maximum principle, and the local mapping of analytic functions. Cauchy's theorem is then proved under more general conditions, and the chapter on complex integration ends with a discussion of the calculus of residues.

The topics which are now discussed include infinite products and entire functions through Hadamard's factorization theorem, Harnack's principle, the gamma function, Riemann's mapping theorem, and subharmonic functions. The discussion of subharmonic functions is carried far enough to solve the Dirichlet problem for regions of finite connectivity. Harmonic measure is discussed briefly and regions of finite connectivity are mapped onto canonical regions.

The last chapter is concerned with multiple-valued functions. It includes a precise discussion of Riemann surfaces, branch points, homotopic curves, and algebraic functions. The book concludes with a lucid discussion of the hypergeometric function.

Among the topics not included are elliptic functions and, surprisingly in view of the author's own work, the length-area principle and the Phragmén-Lindelöf principle.

The reviewer noted only two places where the proof involves more than the text implies, namely p. 170, line 8 and p. 188, line 18. Each gap can be filled by well known methods but the first is the more serious because the simplest proof (that occurred to the reviewer) depends on ideas developed between pages 170, 188. The book is an important contribution to mathematical literature. At every turn one sees the care and ingenuity which the author has used to make his proofs rigorous and readable. The book is intended for the conscientious student, and it will repay him well for the hours that he may spend with it.

A. C. SCHAEFFER

Foundations of the nonlinear theory of elasticity. By V. V. Novozhilov. Trans. from the first (1948) Russian ed. by F. Bagemihl, H. Komm, and W. Seidel. Rochester, Graylock, 1953. 6+233 pp. $4.00.

Students of mechanics will be grateful to the translators and the publishers for making available the second of the three[1] existing monographs on the general theory of elasticity—the more so, since the Russian original is in this country at least a very rare book.

The translation is unusually good English (except for "compatability") and the translators have taken unusual care that the exposition of this elaborate subject shall make sense, although they are not always familiar with the terms used in mechanics (e.g. on p. 58 they use "components of a vortex" for "components of the curl"). Despite its being planographed, and thus repulsive to the eye, the text is readable.

The author's approach is straightforward, honest, and vigorous. There is little or no nationalism, rhetoric, or pedagogery. The author gives every evidence of his earnest competence and his respect for a difficult and important group of problems. The book is not scholarly, however; most of the some ninety items in the bibliography are not cited in the text, part of which presents material first published in important papers not listed in the bibliography. It is quite possible that many of the results in this book are rediscoveries by the author himself.

This is a serious work, deserving detailed notice. The author's preface is dated 1947, and the book is on the whole a careful, accurate, and reliable exposition of some of the mechanical aspects of the classical nonlinear theory of elasticity as it stood at that date. It was in 1948 that the numerous publications of Rivlin, which have enlivened the subject and changed the whole view of it, began to appear.[2] Thus

[1] The other two are *Théorie des corps déformables* by E. and F. Cosserat, Paris, 1909, and *Finite deformation of an elastic solid* by F. D. Murnaghan, New York, 1952. The latter was reviewed in Bull. Amer. Math. Soc. vol. 58 (1952) pp. 577–579.

[2] These are briefly summarized in Chap. IV of my paper, *The mechanical foundations of elasticity and fluid mechanics*, Journal of Rational Mechanics and Analysis vol. 1 (1952) pp. 125–300; corrections and additions, vol. 2 (1953) pp. 593–616.