on the cubic surface, defined as the locus of intersections of corresponding planes of three related stars, which leads to the determinantal equation and the plane mapping—a treatment closely parallel to Reye's but expressed in algebraic terms. The pencil, range, and net of quadrics are then considered, with the cubic transformations determined by the latter, and Hesse's correspondence between its Jacobian sextic and a plane quartic—the last, only to the point of relating the 28 bitangents of the quartic to the 28 pencils of double contact in the net. Linear transformations in space are then treated, though nothing like a complete enumeration is attempted. There is here an interesting treatment of the collineations which leave a quadric invariant, and the applications of this theory to non-Euclidean geometry.

Last of all come a brief—too brief, perhaps—introduction to line geometry; and a final hint at the possibilities of $n$-dimensional geometry. The workings of duality in four and five dimensions, the general intersections relations in these spaces, the properties of the fifth associated plane in four dimensions, and the representation of conics in the plane and lines in ordinary space by points of five dimensions are offered to whet the appetite; and the student is left, a little abruptly perhaps, mature in outlook, and able to start in earnest on algebraic geometry.

P. DU VAL

*Real functions.* By Casper Goffman. New York, Rinehart, 1953. 12+263 pp. $6.00.

*Principles of mathematical analysis.* By Walter Rudin. New York, McGraw-Hill, 1953. 9+227 pp. $5.00.

*Theory of functions of real variables.* By Henry P. Thielman. New York, Prentice-Hall, 1953. 11+209 pp. $6.65.

These three books, each an introductory text on real function theory, have appeared almost simultaneously. This unusual situation has led the reviewer to write a single article comparing the three rather than to write a separate review of each. Thus, the grouping of the three books into one review is not to be taken as an indication that no one of them is of sufficient significance to merit a separate discussion. Rather, it is a recognition of the fact that they will be considered competitively so that a discussion of their relative merits would seem to be the most pertinent.

The following chart gives a brief summary of the contents of the three books. It lists the major topics considered in the union of the three books and gives under each author's name the reviewer's opinion of that author's treatment of the indicated topic.

| Topic | Goffman | Rudin | Thielman |
|---|---|---|---|
| Real number system | good | good | good |
| Algebra of sets | good | good | good |
| Cardinal numbers | good | good | good |
| Ordinal numbers | good | none | incomplete |
| Elements of topology | | | |
| in Hausdorff spaces | none | none | good |
| in metric spaces | sketchy | sketchy | good |
| in Euclidean $n$-space | none | good | none |
| on real line | good | good | good |
| Limits superior and inferior | very sketchy | good | good |
| Continuity | good | elegant | good |
| Discontinuous functions | good | very sketchy | good |
| Infinite series | none | good | good |
| Fourier series | none | good | none |
| Uniform convergence | good | good | sketchy |
| Differentiation | | | |
| properties of derivatives | sketchy | good | very sketchy |
| existence theorems | very good | none | good |
| partial derivatives | none | good | none |
| Borel sets | good | very sketchy | good |
| Measure | specialized | abbreviated | very good |
| Riemann integrals | good | very good | good |
| Lebesgue integrals | involved | good | ineffectual |
| Differentiation of integrals | complicated | specialized | good |

Some of these points call for more detailed discussion.

Each author obtains the real number system by extending smaller systems. Thielman starts with the positive integers (Peano postulates) and works through several extensions. Each of the others starts with the rational number system. Goffman characterizes it as a minimal ordered field and gives a uniqueness proof. Rudin merely appeals to the reader's intuition to accept the rational number system. Rudin and Thielman make the completeness extension by Dedekind cuts. Goffman does it first by Cauchy sequences, then does it again by Dedekind cuts. His proof of the equivalence of these procedures makes no mention of the fact that this equivalence depends on the Archimedean character of the rational field. He has mentioned informally (p. 30) that the rational field is Archimedean, but he has not proved it or even stated it as a theorem. Similarly, Rudin (p. 39) makes an informal statement which (while not inaccurate inasmuch as it refers to the real number system) seems to ignore the fact that there are non-Archimedean ordered fields which are Cauchy sequence complete but not Dedekind cut complete.

While on the subject, the reviewer would like to beat the drum for the idea that the discussion of extensions of number systems (involving, as it does, principally a verification of algebraic postulates in the extended system) belongs in an algebra course. For purposes of studying analysis it should suffice to describe the real number system as a complete ordered field and prove (or, preferably, refer to a proof of) uniqueness. Furthermore, the existence of least upper bounds is probably the most efficient completeness postulate—efficient in the sense of being directly applicable to the most theorems in real function theory.

Rudin avoids transfinite arguments altogether. Thielman mentions the linear ordering of the cardinals and develops the theory of ordinals to the point where he needs only the well-ordering theorem to prove linear ordering. He then states the well-ordering theorem informally, chooses not to prove it, and fails to point out its implications. Goffman gives a fairly complete discussion of ordinals including the amazing phenomenon of a proof of the well-ordering theorem with no mention of any transfinite axiom. The axiom of choice appears in the first paragraph of his proof in the following interesting way: "Let $S$ be a set. Let there be a function $f(A)$ which associates an element $a \in S-A$ with every proper subset $A \subset S$, including the empty set." Perhaps this unusual wording makes it an axiom rather than a supposedly obvious statement; however it sounds more like the divine pronouncement, "Let there be light," in the first chapter of Genesis.

For an efficient discussion of uniform convergence and the reversal of order in iterated limits, the reviewer suggests the following theorems as a basis: (1) If both one-variable limits of $f(x, y)$ exist pointwise in the neighborhood of a point and if one of them is uniform (sometimes called subuniform) at the point, then the double limit exists at the point. (2) If both one-variable limits of $f(x, y)$ exist pointwise in the neighborhood of a point and if the double limit exists at this point, then (a) the iterated limits exist and are equal at this point, and (b) both one-variable limits are uniform at this point. (3) For a pointwise convergent sequence of continuous functions the set of points at which the convergence is not uniform is an $F_\sigma$ set of the first category. Proofs of these theorems are not particularly long, and the immediate corollaries are too numerous to mention.

Thielman proves only that the uniform limit of a sequence of continuous functions is continuous. Rudin does not discuss category; so he is not interested in (3) above. He does prove (1) and (2a) with the stronger hypothesis in (1) that one of the one-variable limits is uniform on some fixed neighborhood of the point in question. He then makes very effective use of this result to get easy proofs of many corollaries, including that on order of summation of absolutely convergent double series. Goffman introduces the notions of category and of uniformity at a point and proves most of the interesting corollaries to theorems (1)–(3) above. However, he never actually states any one of the general theorems (1)–(3), with the result that his discussion is more circuitous than it needs to be.

Goffman's discussion of measure is limited to a direct development of Lebesgue measure on [0, 1]. Rudin discusses Lebesgue measure in Euclidean $n$-space, but he does it by abstract measure theory methods. That is, he discusses the extension of a completely additive function from a ring to a $\sigma$-ring. In describing the ring of elementary figures, he refuses to specify the closure properties of his intervals; so it is not at all clear that he has a ring. In defining an outer measure, he covers sets by intervals without specifying what kind. Later (p. 198) in order to prove regularity of the measure, he says that these covering intervals may be assumed to be open without affecting the value of the outer measure. This is true for continuous measures, therefore true for his only concrete example; but he pretends not to be restricting himself to this case. Rudin's whole discussion of measure, while elegant in outline, is sketchy in development and (as indicated above) somewhat careless. Thus, the last chapter tends to mar an otherwise beautifully written book. Thielman's discussion of measure is the best of the three. He develops Lebesgue measure in Euclidean $n$-space by the Carathéodory method of working with a postulationally defined outer measure.

Rudin defines the Lebesgue integral by means of approximations from below by simple functions. Goffman uses Fréchet's method of forming infinite series by considering infinite subdivisions of the entire range space. This leads to rather complicated looking proofs of the properties of the integral. Also, it restricts the discussion to domains with finite measure. Thielman uses Lebesgue's method of considering finite subdivisions of the range for bounded functions and then extending to the unbounded case by means of truncated functions. He then does it all over again by considering finite subdivisions of the domain into measurable subsets. For some unexplained reason this latter procedure is supposed to be particularly significant in connection with integrals defined over Borel sets. If there is a point to Thielman's discussion of integrals over Borel sets, the reviewer missed it.

Thielman proves practically none of the important properties of the integral, not even complete additivity. Only Goffman and Rudin prove the Lebesgue dominated convergence theorem. Only Goffman proves Egoroff's theorem, and only Rudin proves the Riesz-Fischer theorem. None of the three considers convergence in measure or gives necessary and sufficient conditions for mean convergence.

Rudin considers differentiation of integrals only for Riemann integrals of continuous functions. Goffman and Thielman both develop the Lebesgue theory. Thielman does a very neat job of it. Goffman makes it hard for himself by failing to prove and use the lemma that if $\overline{D}F(x) \geqq a$ almost everywhere in $E$, then $F(E) \geqq am(E)$.

Only Goffman and Thielman discuss Fubini's theorem. They are both rather vague about extending the proof from the case of nonnegative functions to the general case. Neither points out the difficulty that if $\phi(y) = \int_X f(x, y)dx$, then it is not necessarily true that $\phi^+(y) = \int_X f^+(x, y)dx$. In fact, Thielman's statement (p. 180) that all his preceding results can be extended to functions with variable sign is incorrect.

Rudin is consistent in writing $f$ for a function and $f(x)$ for one of its values. However, he slips up on sequences. He says (p. 19), "By a sequence we mean the values of a function $f$ defined on the set $J$ of all positive integers." Does this mean a sequence is a point set? Goffman speaks of a mapping $f$ and a function $f(x)$. He says a sequence is a mapping and explains that the sequence $\{x_i\}$ and the point set $\{x_i\}$ are two different things. The notation is unfortunate, to say the least. Thielman is sound on the subject of sequences, but he varies from one place to another in his use of $f$ and $f(x)$. Everyone seems to have trouble defining a series. Goffman gets out the easy way; he does not discuss series. Rudin does not commit himself to a precise definition, but he has a succession of remarks (p. 44) from which can be inferred first that the symbol $\sum_{n=1}^{\infty} a_n$ stands for a sequence (the sequence of partial sums), then that it stands for a number (the sum of the series). Thielman defines a series as a sequence (of partial sums) and then works himself into a verbal corner involving a series whose terms are its own partial sums!

Finally, an over-all view: Rudin's book is definitely the smoothest. He lists his theorems in the most effective order for facilitating his arguments, and he invariably comes up with extremely neat proofs. However, he does not push his investigations quite as far as the other authors do. A notable example of this is his failure to follow an excellent discussion of continuous functions by an investigation of category and Borel sets and their connection with real functions. With any topic he considers, Goffman pushes farther than either of the other two authors. In this sense, his book is the most complete of the three. Unfortunately, too many of his proofs are ungainly and complicated. Essentially, this stems from ineffective organization. In some places (compare his development of topology on the real line with Rudin's, for example) proofs could be simplified by rearranging the theorems. In other places (note the reviewer's comment on his treatment of differentiation of Lebesgue integrals) he repeats an argument several times because he fails to pull out an intermediate result which could be reused. Thielman falls in between. His proofs are not particularly striking, but standard and reasonably efficient and uncomplicated. He does not dig as deeply as Goffman, but more deeply than Rudin. The blight on Thielman's book is the surprising number of inaccurate statements and incomplete proofs. On a first reading the reviewer found ten examples, each of a flaw of one of the following types: informal statements of results that are not true, failure to consider all cases in a proof, use of concepts which have not been defined, use of results either without proof or before they have been proved. One such example was mentioned in the discussion of Fubini's theorem above. To cite one other, Thielman's proof that the union of a denumerable set of denumerable sets is denumerable tacitly assumes that the sets are disjoint.

Each of the three authors writes a very pleasing line of prose; so there is very little choice to be made on that score. Also, each book has an adequate supply of worthwhile exercises. The reviewer is unable to rate one above the other on this point.

There are a few typographical errors in each book. Those worth mentioning involve errors in cross references: Rudin—p. 74, line 22, for 4.1 read 4.2. Thielman—p. 97, line 6, for 3.2.2 read 3.2.1. Thielman—p. 184, line 7 from bottom, for 9.8.1 read 9.11.1.

M. E. MUNROE

*Numerical solution of differential equations.* By W. E. Milne. New York, Wiley, 1953. 11+275 pp. $6.50.

This book contains the first general treatment, in English, of numerical methods for solving differential equations. The author has been able to cover in the 275 pages only those classes of problems and methods which he considers most important. The methods are presented very clearly, with completely worked numerical examples, and should be easily mastered by the average reader. On the other hand it is evident from the choice of methods, particularly for problems involving latent roots of matrices and elliptic differential equations,