curve theorem, which to the author's knowledge is never needed in function theory."

What is the place which Thron's book will occupy in the literature? Certainly it contains much valuable material, well organized and in convenient form for coordinated study. For this reason it belongs (a) in the library of every college which makes an attempt to teach mathematics, and (b) in the personal library of every specialist in function theory. However, just because it is so carefully written, with so much attention devoted to foundations, it should never (in the reviewer's opinion) be used as text either in a beginning or advanced course in function theory. The author is to be congratulated for his courage in writing such a book and his success in finding a publisher, and the publisher in turn has performed a real service for mathematics.

A. W. GOODMAN

*Isoperimetric inequalities in mathematical physics.* By G. Pólya and G. Szegö. (Annals of Mathematics Studies, no. 27.) Princeton University Press, 1951. 16+279 pp. $3.00.

The title of this book, as remarked by the authors in the preface (where the authors have admirably delineated the aims of the present work), suggests its connection with a classical subject of mathematical research, the "isoperimetric problem." This problem consists in seeking among all closed plane curves, without double points and having a given perimeter, the curve enclosing the largest area. The "isoperimetric theorem" gives the solution to the problem: of all curves with a given perimeter, the circle encloses the maximum area. If the perimeter of a curve is known, but the exact value of its enclosed area is not, the isoperimetric theorem yields a modicum of information about the area, an upper bound, an "isoperimetric inequality"; the area is not larger than the area of the circle with the given perimeter. There are, besides perimeter and area, many important geometrical and physical quantities (set functions, functionals) which depend upon the size and shape of a curve. There are many inequalities, similar to the isoperimetric inequality, which relate these quantities to each other. By extension, all these inequalities can be called "isoperimetric inequalities." Besides, there are analogous inequalities dealing with solids, pairs of curves (condenser, hollow beam), pairs of surfaces, and so forth. The present book is concerned with inequalities of this type.

An example of such an isoperimetric inequality, with which the subject matter of the book may be said to have begun, is the conjecture of B. de Saint-Venant (1856) concerning the torsion of elastic prisms: of all simply connected cross-sections with a given area, the circle has the maximum torsional rigidity. Inequalities of this sort are of practical value. Consider the inequality: of all triangular membranes with a given area the equilateral triangle has the lowest principal frequency. This furnishes a lower bound for the principal frequency of an arbitrary triangle in terms of its area. In doing this, a not readily accessible physical quantity (the principal frequency) is estimated in terms of easily accessible geometrical data (area, triangular shape). This illustrates the general trend of the present book: to estimate physical quantities on the basis of geometrical data, less accessible quantities in terms of more accessible ones.

This book is intended as a fairly complete account of investigations carried out by the authors for a number of years, and to which many scientists have contributed their efforts. The authors have succeeded in their task of achieving a unified, concise, easily readable presentation of the diversified material. In the course of the present review, references will be given only to papers not quoted in the book. Most of these have appeared since the publication of the book, which is a sign of the current interest in the subject matter under discussion.

A brief preface, which sets the stage for the rest of the book and presents concisely the authors' aims and point of view, has been summarized above. Chapter I, entitled "Definitions, methods and results," is a remarkably clear road map to the remainder of the book. As its title implies, this chapter fixes the notation to be employed throughout; it outlines the chief methods of attack employed: symmetrization, minimum principles, and expansion and variational methods; and it contains the precise statement of many of the results proved later. A great many results are compactly presented in the form of tables. The chapter ends with a brief survey of the following chapters.

Chapters II, III, and IV are concerned with the subject of capacity, and related topics. Chapter II is entitled "The principles of Dirichlet and Thomson." Let $A$ be a closed surface, and $u(p)$ be a function harmonic outside $A$, assuming the constant value $u(p)=u_0$ on $A$ and having the expansion $u(p)=X_0r^{-1}+X_1r^{-2}+X_2r^{-3}\cdots$ near infinity, where $X_0, X_1, X_2, \cdots$ denote surface harmonics of degrees $0, 1, 2, \cdots$ respectively. The ratio $X_0/u_0=C$ depends only on the "conductor" $A$ and is called its capacity. Clearly,

$$C = -\frac{1}{4\pi u_0}\iint \frac{\partial u}{\partial n_e}\,d\sigma = \frac{1}{4\pi u_0^2}\iiint |\operatorname{grad} u|^2 d\tau,$$

where the surface integral is taken over an arbitrary surface enclosing $A$ (possibly $A$ itself), $d\sigma$ and $d\tau$ are the surface and volume elements, respectively, and $n_e$ is the outer normal. The principal purpose of the chapter is to find upper and lower bounds for $C$ in terms of certain geometrical quantities associated with $A$. Upper bounds for $C$ can be found by means of the principle of Dirichlet, which is formulated by the authors as follows: If $f(p)$ is an arbitrary scalar function defined on and outside $A$, with $f=u_0$ on $A$ and $f=0$ at infinity, then

$$C \leqq (4\pi)^{-1}u_0{}^2 \iiint |\operatorname{grad} f|^2 d\tau \qquad \text{(Dirichlet's principle).}$$

Lower bounds for $C$ can be found by means of Thomson's principle, which is formulated by the authors as follows: If $\boldsymbol{f}(p)$ is an arbitrary sourceless vector function defined on and outside $A$, that is,

$$\operatorname{div} \boldsymbol{f}(p) = 0, \qquad \text{outside } A,$$

and

$$(4\pi)^{-1} \iint \boldsymbol{f}\cdot \boldsymbol{n}_e d\sigma = Q,$$

where $Q$ is the total charge, then

$$1/C \leqq (4\pi)^{-1}Q^{-2} \iiint |\boldsymbol{f}|^2 d\tau \qquad \text{(Thomson's principle).}$$

It is clear that by choosing, in the above inequalities, particular functions $f(p)$ and $\boldsymbol{f}(p)$ satisfying the required conditions, one obtains upper and lower bounds for the capacity $C$ of $A$. Direct formal proofs for the above inequalities are given, and it is shown that a certain minimum principle due to Gauss follows as a special case of Thomson's principle. The reviewer, however, has a predilection for the following derivation (patterned after J. B. Diaz and A. Weinstein, Journal of Mathematics and Physics vol. 26 (1947) pp. 133–136) which is based directly on Schwarz's inequality and clarifies the interrelation between the various minimum principles. For simplicity in writing, take $u_0=1$; then the problem of estimating $C$ is that of finding upper and lower bounds for the Dirichlet integral

$$\iiint |\operatorname{grad} u|^2 d\tau \qquad (=4\pi C)$$

of the solution $u$ of the Dirichlet problem:

$$u_{xx} + u_{yy} + u_{zz} = 0, \qquad \text{outside } A,$$
$$u = 1, \qquad \text{on } A,$$
$$\lim_{p \to \infty} u(p) = 0.$$

Schwarz's inequality states that

$$\left[\iiint \phi \cdot \psi d\tau\right]^2 \leqq \left(\iiint \phi \cdot \phi d\tau\right)\left(\iiint \psi \cdot \psi d\tau\right),$$

the integrals being taken over the exterior of $A$. Let $f(p)$ be an arbitrary scalar function defined on and outside $A$, with $f=1$ on $A$ and $f=0$ at infinity, and take $\phi = \operatorname{grad} f$, $\psi = \operatorname{grad} u$ in Schwarz's inequality. An application of Green's identity readily yields Dirichlet's principle

$$\iiint |\operatorname{grad} u|^2 d\tau \leqq \iiint |\operatorname{grad} f|^2 d\tau.$$

Now let $\boldsymbol{f}(p)$ be an arbitrary (nonzero) vector function defined on and outside $A$, with $\operatorname{div} \boldsymbol{f}(p) = 0$, outside $A$, and take $\phi = \boldsymbol{f}$, $\psi = \operatorname{grad} u$ in Schwarz's inequality. An application of Green's identity

$$\iiint \boldsymbol{f} \cdot \operatorname{grad} u d\tau = \iiint [\operatorname{div}(u\boldsymbol{f}) - u \operatorname{div} \boldsymbol{f}] d\tau = -\iint u\boldsymbol{f} \cdot \boldsymbol{n}_e d\sigma$$

readily yields an inequality equivalent to Thomson's principle

$$\frac{\left(\iint \boldsymbol{f} \cdot \boldsymbol{n}_e d\sigma\right)^2}{\iiint |\boldsymbol{f}|^2 d\tau} \leqq \iiint |\operatorname{grad} u|^2 d\tau.$$

If, in particular, $\boldsymbol{f} = \operatorname{grad} v$, where $v$ is a nonconstant harmonic function, the last inequality yields (for the special Dirichlet problem under consideration here) E. Trefftz's (Verhandlungen Congress für Technische Mechanik, Zürich, 1927, p. 131) lower bound for the Dirichlet integral of the solution of Dirichlet's problem:

$$\frac{\left(\iint \frac{\partial v}{\partial n_e} d\sigma\right)^2}{\iiint |\operatorname{grad} v|^2 d\tau} \leqq \iiint |\operatorname{grad} u|^2 d\tau.$$

If, in particular, the harmonic function $v$ is given outside $A$ by a potential of a single layer distribution, with density $\mu$, over $A$, that is

$$v(p) = \iint \mu(q) \frac{1}{r_{pq}} d\sigma_q$$

(which serves to define $v$ throughout space) then the last inequality (using the known jump condition for the normal derivative of a single layer potential) may be written

$$\frac{\left(\iint \frac{\partial v}{\partial n_e} d\sigma\right)^2}{4\pi \iint v\mu d\sigma - \iiint_{\text{interior of } A} |\operatorname{grad} v|^2 d\tau} \leqq \iiint |\operatorname{grad} u|^2 d\tau;$$

and, a fortiori, using Green's theorem to simplify the numerator,

$$4\pi\left(\iint \mu d\sigma\right)^2 \Big/ \iint v\mu d\sigma \leqq \iiint |\operatorname{grad} u|^2 d\tau,$$

which is equivalent to the principle of Gauss referred to above.

In order to make actual use of the inequality of Dirichlet's principle, suitable functions $f(p)$ must be chosen. The authors propose to choose first the level surfaces of $f$ (as indicated in chapter I there are intuitive reasons of various kinds for preferring certain families of level surfaces) and then to find the "best function" possessing the prescribed level surfaces. Let $\psi(p) = \nu$ be the equation of the given level surfaces, call them $A(\nu)$, where $0 \leqq \nu < \infty$, and $A(0) = A$. Any function $f(p)$ having these prescribed level surfaces must be of the form $f(p) = \lambda(\psi(p))$ where $\lambda(t)$ is defined on $0 \leqq t < \infty$, and $\lambda(0) = u_0$, $\lambda(\infty) = 0$. It is shown that if

$$T(\nu) = \frac{1}{4\pi} \iint_{A(\nu)} |\operatorname{grad} \psi| \, d\sigma,$$

which depends only on the given surfaces $A(\nu)$, then upon choosing

$$\lambda(t) = \int_t^\infty [T(\nu)]^{-1} d\nu$$

one has

$$C \leqq \frac{1}{\int_0^\infty [T(\nu)]^{-1} d\nu},$$

which is the best upper bound for $C$ obtainable from Dirichlet's principle once the level surfaces $A(\nu)$ are prescribed. In order to make actual use of the inequality of Thomson's principle, vector functions $f(p)$ must be chosen. Again, there are intuitive reasons for choosing a certain set of *lines of force* first. These lines do not determine $f(p)$ completely, but only its direction at each point $p$. The authors determine the best lower bound for $C$ obtainable from Thomson's principle once the lines of force are prescribed in a certain way. A method of approximation given for the capacity is based on the theorem (when $A$ is analytic, or, if not analytic, is star shaped with respect to an interior point $p_0$) that $1/C$ is the greatest lower bound of the quantities

$$-\frac{1}{4\pi}\int\int_A V_0 \frac{\partial V_0}{\partial n_e} d\sigma = \frac{1}{4\pi}\int\int\int_{\text{exterior of } A} |\operatorname{grad} V_0|^2 d\tau$$

(compare with Trefftz's lower bound for $C$ given above) where $V_0$ is an arbitrary harmonic function of the special type

$$V_0 = r^{-1} + H_1(x, y, z)r^{-3} + H_2(x, y, z)r^{-5} + \cdots + H_n(x, y, z)r^{-2n-1},$$

with $r$ the distance from a fixed point $p_0$ in the interior of $A$, and $H_n(x, y, z)$ the most general homogeneous polynomial of degree $n$ in $x, y, z$ satisfying Laplace's equation. The principles formulated above are shown to apply to other charge distributions in which one has to deal with a pair of surfaces (a "condenser"). The analogous problems in two dimensions are also treated.

Chapter III bears the title: "Applications of the principles of Dirichlet and Thomson to estimation of the capacity." These principles are applied mainly in the form indicated in chapter II by choosing the level surfaces and the lines of force appropriately. Three classes of surfaces are considered: (1) convex surfaces, (2) surfaces which are star shaped with respect to a certain interior point $p_0$, and (3) surfaces of revolution. In (1) the exterior parallel surfaces and the normals to the given surface are chosen; in (2) the surfaces similar (with respect to $p_0$) to the given surface and the rays issuing from $p_0$ are chosen; while in (3) the surfaces of revolution obtained by rotating the level curves of the exterior conformal mapping of the meridian curve of the given surface onto a circle, and the curves in this mapping which correspond to the radii of the circle, are employed. As an example of the results obtained, consider the capacity $C$ of a surface of revolution, representing the meridian curve in a complex $w$-plane and choosing the real axis as axis of symmetry. Let $w = f(z)$ be the mapping function of the exterior of this curve onto the exterior of the circle $|z| = \bar{r}$:

$$w = f(z) = z + c_0 + c_1 z^{-1} + \cdots + c_n z^{-n} + \cdots .$$

Then

$$\int_0^{\pi} \left\{ \int_{\bar{r}}^{\infty} [u(r, \theta)]^{-1} dr \right\}^{-1} d\theta \leqq C \leqq \left\{ \int_{\bar{r}}^{\infty} \left[ \int_0^{\pi} u(r, \theta) d\theta \right]^{-1} dr \right\}^{-1},$$

where

$$r \operatorname{Im} f(re^{i\theta}) = 2u(r, \theta).$$

The equality signs hold for spheroids and only for spheroids. The same chapter also contains a proof of the Poincaré-Faber-Szegö inequality

$$C \geqq \left( \frac{3V}{4\pi} \right)^{1/3},$$

where $C$ is the capacity of a closed surface and $V$ is the volume of the solid bounded by the surface. The proof depends upon a combination of Dirichlet's principle and the process of symmetrization with respect to a point. As a particular, interesting special case of the inequalities obtained in this chapter, the following inequalities for the capacity $C$ of a cube with edge $a$ are given:

$$0.632a < C < 0.71055a.$$

Chapter IV is entitled "Circular plate condenser." A circular plate condenser consists of two congruent circular disks $A_0$ and $A_1$ with a common axis. Let the common radius be denoted by $a$, and the distance between the planes of the circular disks be $c$. Let $Q > 0$ and suppose $Q$ and $-Q$ are the total charges of $A_0$ and $A_1$ respectively. In the position of equilibrium the potential will be constant on the disk and equal to $\pm V_0$. The problem of estimating the capacity $C = Q/2V_0$ of the condenser for small values of the ratio $c/a = q$ had been considered earlier by Kirchhoff and Ignatowsky. Using Gauss' principle, the authors obtain the following result:

$$\frac{C}{a} > \frac{1}{4q} + \frac{1}{4\pi} \log \frac{1}{q} + \frac{1}{4\pi}\left( \log 8 - \frac{1}{2} \right) + \epsilon,$$

where $\epsilon \to 0$ as $q \to 0$. The principle of Gauss is applied by assuming uniform charges of equal magnitude and opposite signs over the disks $A_0$ and $A_1$. A generalization of this line of reasoning, assuming an arbitrary circular-symmetrical charge distribution, leads not only to other lower bounds for $C$, but also, at least theoretically, to an exact formula for $C$. As a special case of the estimation of the capacity of a condenser consisting of two solids arising from each other by reflection in a plane, the chapter concludes with the following estimate for $C$ for a circular plate condenser when $c$ is large:

$$C = \frac{a}{\pi}\left(1 + \frac{2a}{\pi c}\right) + O\left(\frac{1}{q^2}\right).$$

Chapter V, entitled "Torsional rigidity and principal frequency," is divided into three main parts. The first part contains variational definitions for the torsional rigidity and the principal frequency, and consequences of these definitions. Let $D$ be a bounded simply connected plane domain, $C$ be its boundary curve, $P$ its torsional rigidity, and $\Lambda$ its principal frequency. Usually, the torsional rigidity of $D$ is defined by the equation

$$P = 2\int\int_D v\,dxdy$$

where the function $v$ (the stress function) is the solution of the following boundary value problem:

$$v_{xx} + v_{yy} + 2 = 0, \qquad \text{in } D;$$

$$v = 0, \qquad \text{on } C.$$

The variational definition for $P$ used in this book is contained in the following inequality:

$$\frac{\int\int_D (f_x^2 + f_y^2)dxdy}{4\left(\int\int_D f dxdy\right)^2} \geqq \frac{1}{P},$$

where $f$ is a sufficiently smooth, not identically zero, real-valued function defined on $D+C$, satisfying the boundary condition $f=0$ on $C$. The equality sign holds if and only if $f=cv$, for some real number $c\neq 0$, where $v$ is the stress function. (It may be remarked at this point that the reviewer has indicated, in the Proceedings of the Symposium on Spectral Theory and Differential Problems, Oklahoma A. and M. 1951, pp. 279–289, another variational definition for $P$, which is contained in the inequality

$$P \leqq \int\int_D (g_x^2 + g_y^2)dxdy$$

where $g$ is a sufficiently smooth function defined on $D+C$, satisfying the partial differential equation $g_{xx}+g_{yy}=-2$ in $D$. Here the "arbitrary" function $g$ satisfies the same partial differential equation as the stress function $v$, and furnishes an upper bound for $P$, whereas in the authors' principle, the arbitrary function $f$ satisfies the same boundary condition as the stress function $v$ and yields a lower bound for $P$. This principle, which is related to Trefftz's lower bound (mentioned before) for the Dirichlet integral of the solution of Dirichlet's problem in terms of an arbitrary harmonic function, has been extended to the case of multiply connected domains and applied to the estimation of the torsional rigidity by H. F. Weinberger, Journal of Mathematics and Physics vol. 32 (1953) pp. 54–63.) The variational definition for $\Lambda$ used in this book is contained in the inequality

$$\frac{\iint_D (f_x^2 + f_y^2)dxdy}{\iint_D f^2 dxdy} \geqq \Lambda^2,$$

where the arbitrary function $f$ is as indicated above. The equality sign holds if and only if $f=cw$, for some real number $c\neq 0$, where $w$ is the solution of the following boundary value problem:

$$w_{xx} + w_{yy} + \Lambda^2 w = 0, \qquad w > 0, \qquad \text{in } D,$$
$$w = 0, \qquad \text{on } C.$$

The variational definitions, together with Schwarz's inequality, are shown to yield the following interesting inequality connecting $P$, $\Lambda$ and the area $A$ of $D$:

$$P\Lambda^2 > 4A.$$

Assuming that $C$ is star shaped, the method of choosing $f$ having level lines which are similar to $C$ is employed to obtain upper bounds for $1/P$ and $\Lambda^2$, in terms of an arbitrary function, which serves the purpose of assigning the values of $f$ on the already prescribed level lines. Ingenious choices of this last arbitrary function lead to many interesting inequalities. The theory of conformal mapping is also employed to obtain further inequalities for $P$ and $\Lambda$, for example $P\geqq(\pi/2)\dot{r}^4$, where $\dot{r}$ is the maximum inner radius of $D$, the equality sign holding only when $D$ is a circle. A more general treatment of the problem of finding lower bounds for $P$ is also given, when a set of curves is prescribed as the level curves of the arbitrary function $f$ appearing in the variational definition of $P$. Let $C_\rho$, where $0\leqq\rho\leqq 1$, denote an arbitrary set of curves, "filling up the domain $D$," in such a way that $C_0$ is an interior point (or a finite set of interior points) of the domain $D$, $C_1$ is the boundary $C$ of $D$, and $C_\rho$, for $0<\rho<1$, is a simple closed curve (or a finite set of mutually exclusive simple closed curves) such that $C_\rho$ lies in the interior of $C_{\rho'}$ if $\rho<\rho'$. Then

$$P \geqq 4\int_0^1 [A(\rho)]^2[\lambda(\rho)]^{-1}d\rho$$

where $A(\rho)$ is the total area bounded by $C_\rho$ and

$$\lambda(\rho) = \int_{C_\rho} |\operatorname{grad} \rho|\, ds.$$

The second part of chapter V contains the "inclusion lemma" and its application to deduce Saint Venant's approximate formula for torsional rigidity. The term "inclusion lemma" is used by the authors to refer to four lemmas whose content can be intuitively expressed, in the authors' words, as saying that "an arbitrary convex curve is only boundedly different from a suitable rectangle or a suitable ellipse." The application consists in showing that $PIA^{-4}$ (where $I$ is the polar moment of inertia of $D$ with respect to its centroid) is contained between positive bounds for an arbitrary closed, bounded, convex domain $D$. The third part of chapter $V$ contains applications of conformal mapping. Consider the conformal mapping

$$z = a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_n\zeta^n + \cdots, \qquad z = x + iy,$$

which maps $D$ onto the interior of the circle $|\zeta|<1$. If $v(x, y)$ is the stress function, then the function $\Phi$:

$$\Phi(x, y) = v(x, y) + \tfrac{1}{2}(x^2 + y^2)$$

is harmonic in $D$ and is the real part of an analytic function (determined up to a purely imaginary constant)

$$F = \Phi + i\Psi$$

of the complex variable $z=x+iy$, which is regular in $D$. Since $z=z(\zeta)$ is analytic, one has

$$F(z(\zeta)) = u_0 + u_1\zeta + u_2\zeta^2 + \cdots + u_n\zeta^n + \cdots$$

and the problem of determining $\Phi$ (or $v$) is thus equivalent to determining the sequence of coefficients $u_0, u_1, u_2, \cdots$. Expansions for the polar moment of inertia are given in terms of the coefficients $a_n$ of the mapping function $z=z(\zeta)$ and a proof of Saint Venant's theorem (of all simply connected sections with a given area, the circular cross section has the maximum torsional rigidity) is given. This theorem can be expressed by the inequality

$$2\pi P \leqq A^2,$$

where $A$ is, as usual, the area of the cross section. These considerations are based initially on the assumption that the series

$$|a_1| + 2|a_2| + \cdots + n|a_n| + \cdots$$

converges, but this assumption is later relaxed to that of the convergence of

$$|a_1|^2 + 2|a_2|^2 + \cdots + n|a_n|^2 + \cdots .$$

(In connection with the relationship between $I$, the polar moment of inertia of $D$ with respect to its centroid, and the torsional rigidity $P$, the reviewer would like to recall at this point the simple inequality

$$P \leqq I,$$

which was emphasized by the reviewer and A. Weinstein as an application of the formula, valid for simply or multiply connected $D$,

$$P = I - \iint_D (\phi_x^2 + \phi_y^2)dxdy.$$

where $\phi$ is the warping function in torsion.)

Chapter VI is entitled "Nearly circular and nearly spherical domains." The nature of the results contained here is perhaps best illustrated by considering the first example of this kind of investigation, due to Lord Rayleigh. Let $\bar{\rho}(\phi)$ be a fixed real valued function, $0 \leqq \phi \leqq 2\pi$, $\delta$ be a real number (in a sufficiently small neighborhood of zero) and consider the Fourier series expansion

$$\delta\bar{\rho}(\phi) = a_0 + 2\sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi).$$

Rayleigh found that the principal frequency $\Lambda = \Lambda(\bar{\rho}, \delta)$ of the "nearly circular membrane" (the equation of whose boundary in polar coordinates $r$, $\phi$ is $r = 1 + \delta\bar{\rho}(\phi)$) is given by

$$\frac{j}{\Lambda} = 1 + a_0 - \sum_{n=1}^{\infty} \left(1 + \frac{2jJ_n'(j)}{J_n(j)}\right)(a_n^2 + b_n^2) + o(\delta^2)$$

as $\delta \to 0$, where $j$ is the first positive root of the Bessel function $J_0(x)$, $j = 2.4048 \cdots$. Thus, Rayleigh expanded the variation $\delta\bar{\rho}$ of the circular boundary $r=1$ in a Fourier series and expressed the first and second variations of the physical quantity $\Lambda^{-1}$ (dependent on the nearly circular boundary) in terms of the Fourier coefficients. The authors apply this method systematically, extending it by analogy from the plane to space, passing from Fourier expansions to expansions in spherical harmonics. A table in chapter I presents a concise survey of many of the results developed in chapter VI. The first part of the table lists physical quantities $Q$ associated with a nearly circular curve $C$, i.e. expansions of the form

$$Q = 1 + a_0 + \sum_{n=1}^{\infty} R(n)\cdot(a_n^2 + b_n^2)$$

(neglecting terms of higher than second order in $\delta$) where the Fourier coefficients $a_n$ and $b_n$ are as indicated above, and the $R(n)$ depend on the physical quantity $Q$. The second part of the table lists expansions of the form

$$Q = 1 + X_0 + \sum_{n=1}^{\infty} R(n)\cdot\frac{1}{4\pi}\iint [X_n(\theta, \phi)]^2 d\omega,$$

for a nearly spherical surface whose equation, in spherical coordinates $r, \theta, \phi$, is

$$r = 1 + \delta\bar{\rho}(\theta, \phi),$$

with $\bar{\rho}(\theta, \phi)$ a fixed real valued function defined on the unit sphere, $\delta$ a real number in a sufficiently small neighborhood of zero, and

$$\delta\bar{\rho}(\theta, \phi) = \sum_{n=0}^{\infty} X_n(\theta, \phi)$$

the expansion of $\delta\bar{\rho}$ in spherical surface harmonics $X_n(\theta, \phi)$.

Chapter VII is entitled "On symmetrization." In the authors' own words, "the apparently scattered remarks of the present chapter are, in fact, carefully grouped around the idea of symmetrization." At the outset, the idea of symmetrization is connected with two general concepts (similar order and equimeasurability) concerning two real-valued functions of the $n$ real variables $x_1, \cdots, x_n$. In terms of these two concepts, three kinds of symmetrization are defined in $x$, $y$, $z$ space: with respect to a plane (Steiner), with respect to a straight line (Schwarz), and with respect to a point. Numerous results concerning the influence of symmetrization on many physical quantities follow. (L. E. Payne and A. Weinstein, Pacific Journal of Mathematics vol. 2 (1952) pp. 633–641, have recently obtained similar results for a generalized symmetrization which includes Steiner's and Schwarz's as special cases.) To mention only a few of the chapter's results: Schwarz symmetrization diminishes the polar moment of inertia with respect to the centroid; Schwarz symmetrization diminishes the capacity; of all conducting plates with a given area, the circle has the minimum electrostatic capacity (a conjecture of Lord Rayleigh). Many of the results obtained appear to be entirely inacessible to methods other than symmetrization. For example: of all tetrahedra with a given volume, the regular tetrahedron has the minimum surface area, integral of mean curvature, and capacity.

Chapter VIII is entitled "On ellipsoid and lens." The shape of an ellipsoid is characterized by its semi-axes $a$, $b$, $c$, with $a \geqq b \geqq c \geqq 0$. Let $\alpha$, $\beta$, $\gamma$ be the eccentricities of the three principal sections of the ellipsoid, i.e.

$$1 - \alpha^2 = c^2b^{-2}, \qquad 1 - \beta^2 = c^2a^{-2}, \qquad 1 - \gamma^2 = b^2a^{-2},$$

which satisfy the relation

$$1 - \beta^2 = (1 - \alpha^2)(1 - \gamma^2).$$

Approximating the electrostatic capacity by various geometric quantities is one of the principal aims. Let $C'$ be an approximation to the capacity $C$ of the ellipsoid. The relative error $(C' - C)/C$, which is a function of $a$, $b$, $c$, may be expanded in a power series about $(1, 1, 1)$, valid for small $\alpha$, $\beta$, $\gamma$ (i.e. for almost spherical ellipsoids). Written as an expansion in powers of $\beta$ and $\gamma$, this expansion is a sum of homogeneous polynomials of different degrees; the non-identically vanishing polynomial of lowest degree is called the initial term of the relative error of the approximation $C'$. A table in Chapter I lists eight different approximations to $C$ and their corresponding initial terms. In chapter VIII it is shown that one of these approximations:

$$\{11[a + b + c] + 4[(bc)^{1/2} + (ca)^{1/2} + (ab)^{1/2}]\}/45,$$

yields too large values for prolate spheroids and too small values for oblate spheroids; and the same holds for another approximation

$$\left[\frac{M}{4\pi} + \left(\frac{3V}{4\pi}\right)^{1/3}\right] \Big/ 2.$$

The authors illustrate very clearly in chapter I how the study of such particular examples as the ellipsoid and the lens could be useful in a continuation of the present study to lead eventually to a complete system of inequalities between the quantities $C$, $V$, $S$, $M$, $\cdots$.

The book concludes with seven notes and tables for some set functions of plane domains. The tables list (among others) the length $L$, the area $A$, the polar moment of inertia with respect to the centroid $I$, the maximum inner radius $\dot{r}$, the outer radius $\bar{r}$, the torsional rigidity $P$, and the principal frequency $\Lambda$ for a circle, ellipse, narrow ellipse, square, rectangle, narrow rectangle, semicircle, sector, narrow sector, equilateral triangle, and regular hexagon. The notes contain new material and survey the more important contributions to the subject obtained by the authors, sometimes in collaboration with others, up to the date of appearance of the book. (For a recent, related result see G. Pólya, Journal of Mathematics and Physics vol. 31 (1952) pp. 55–57.) Their titles clearly indicate their connection with the various chapters described above. Note A, "Surface-area and Dirichlet's integral," deals with Steiner, Schwarz, and circular symmetrization and their effect on the volume, surface area, Dirichlet integral, and other quantities. Note B, "On continuous symmetrization," deals with the question of defining a transformation $T_\lambda$, depending continuously on a real parameter $\lambda$, with $0 \leqq \lambda \leqq 1$, such that $T_0$ is the identity, $T_1$ is Steiner symmetrization of a plane curve $C$, while $T_\lambda$, for $0 < \lambda < 1$, changes certain quantities associated with the curve $C$ "in the same manner" as is done by Steiner symmetrization of $C$. Note C, "On spherical symmetrization," is concerned with the effect of this symmetrization, which is the three-dimensional analogue of the circular symmetrization of note A, on the capacity of a solid. Note D, "On a generalization of Dirichlet's integral," treats the generalized capacity defined by minimizing the integral

$$\iint_D \left\{ |\operatorname{grad} u|^2 + p(x, y)u^2 \right\} dxdy,$$

where $p(x, y)$ is a given function defined on $D$ and the admissible functions $u$ assume given values on the boundary of $D$. The classical Dirichlet principle arises if $p(x, y) \equiv 0$. Note E, "Heat conduction on a surface," considers the Dirichlet integral

$$\iint_D |\operatorname{grad} f|^2 d\sigma$$

taken over a connected domain of an open or closed surface in three-dimensional Euclidean space, $f$ being a function defined on the surface and the gradient being defined in the sense of the metric of the surface. Lower bounds for the conductance of a "ring shaped" domain $D$ on the surface are found in terms of geometrical quantities connected with $D$. Note F, "On membranes and plates" deals with three quantities $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, defined by the following variational problems:

$$\Lambda_1^2 = \min \frac{\iint |\operatorname{grad} u|^2 d\sigma}{\iint u^2 d\sigma}, \qquad u = 0 \text{ on } C,$$

$$\Lambda_2^4 = \min \frac{\iint (\nabla^2 u)^2 d\sigma}{\iint u^2 d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C,$$

$$\Lambda_3^2 = \min \frac{\iint (\nabla^2 u)^2 d\sigma}{\iint |\operatorname{grad} u|^2 d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C,$$

where $D$ is a bounded plane domain with a simple analytic boundary curve $C$. $\Lambda_1$ and $\Lambda_2$ are the principal frequencies of a membrane with fixed boundary and of a clamped plate, respectively. $\Lambda_3$ occurs in the study of the buckling of plates. Lord Rayleigh formulated the following conjecture (first proved by G. Faber and E. Krahn): of all membranes of a given area the circle has the gravest fundamental tone (lowest principal frequency). This note deals with the analogous problem for $\Lambda_2$ and $\Lambda_3$, under the hypothesis that the functions $u$ for which the minima are attained never vanish in $D$. Note G, "Virtual mass and polarization," is dedicated to relating the quantities in the title to certain geometrical data of a solid.

This review only gives an idea of the nature and the variety of the problems discussed by the eminent authors, whose scientific accomplishments and lucid methods of presentation are well known to the mathematical public. In the brief time since its appearance this book has already become the standard reference text for workers in this field.

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