BOOK REVIEWS

Typical means. By K. Chandrasekharan and S. Minakshisundaram. Oxford University Press, 1952. 10+140 pp. $6.50.

This book, which is the first of a new series of monographs to be published under the auspices of the Tata Institute of Fundamental Research, Bombay, is concerned entirely with the theory and applications of Riesz summability. In view of the fact that Hardy's *Divergent series* devoted only a little space to Riesz summability, there was room for another book dealing more fully with this particular method. While reasonably complete, the book is not exhaustive; indeed, an account of everything that is known on the subject would be impossible in a book of this size. However, a useful series of notes at the end of each chapter contains numerous references, and also the statement of certain theorems for which room was not found in the main text. The book collects in a convenient form much material which has hitherto been available only in the original papers, and it should prove of great use to anyone who wants to work in this particular field. It is a pity that its usefulness should be diminished by the errors which occur in it.

In the remaining remarks, we use the following notation (which is in line with that used in the book). If $\{\lambda_n\}$ is any sequence of positive numbers increasing to infinity, and if $\sum a_n$ is any given series, we write

$$(1)\quad A(t) = A_\lambda(t) = \sum_{\lambda_\nu \leqq t} a_\nu;$$

$$(2)\quad A^k(t) = A_\lambda^k(t) = \sum_{\lambda_\nu \leqq t} (t - \lambda_\nu)^k a_\nu = k\int_0^t (t-\tau)^{k-1} A(\tau)d\tau \quad (k > 0);$$

$A_\lambda^k(t)$ is the Riesz sum, and $t^{-k}A_\lambda^k(t)$ the Riesz mean, of order k and type λ associated with the series $\sum a_n$. If $t^{-k}A_\lambda^k(t) \to s$ as $t \to \infty$, we say that the series is summable (λ, k) to s. [This is more usually termed summability $(R; \lambda, k)$, but as we are concerned only with Riesz summability there is no harm in omitting the "R."]

Apart from some introductory material, Chapter I deals, broadly speaking, with relations between Riesz means of the same type but different orders. Such topics as limitation theorems and M. Riesz's convexity theorem are dealt with adequately. There follows a section (§1.8) on Tauberian theorems. It may be remarked that this section deals only with those Tauberian theorems in which the hypotheses involve only Riesz sums. Those (closely related) Tauberian theorems in which we deduce the convergence or summability (λ, k) of the series $\sum a_n$ from hypotheses about the behaviour of

$$\sum_{n=0}^{\infty} a_n e^{-\lambda_n \sigma} \qquad (\sigma > 0),$$

together with some Tauberian condition are dealt with separately in Chapter III. The treatment of §1.8 has certain shortcomings. Thus in one theorem (Theorem 1.81) one of the hypotheses is that $A(x)$ satisfies a Tauberian condition of the "slowly oscillating" type:

$$A(x) - A(x - t) = O(t^{\gamma} V(x)) \tag{3}$$

for all x and $0 < t < F(x)$, where $\gamma > 0$, and where $V(x)$, $F(x)$ are certain positive functions of x. While the proof is logically valid, the truth of (3) for all x and all sufficiently small t implies that $A(t)$ is continuous, so that if (as appears to be taken for granted throughout the book) $A(t)$ is taken to be of the form (1), the hypotheses can be satisfied only in the trivial case in which $a_n = 0$ for all n. If we take $A(t)$ to be, say, any function which is Lebesgue integrable in any finite interval ($A^k(t)$ then being defined by the integral in (2)) the proof remains valid, and the theorem becomes significant. Perhaps we are intended to take the theorem in this more general sense, but there is no indication of this in the enunciation. Somewhat similar remarks apply to certain other theorems.

Chapter II deals with the "second theorem of consistency," and related matters. Without going into details, the two main theorems in this field can be roughly stated as follows:

(A). Let $\phi(x)$ be a positive function increasing to infinity as $x \to \infty$, which is such that, for some Δ, $\phi(x) = O(x^{\Delta})$, and which satisfies certain other conditions which ensure that, roughly speaking, it should behave in a "reasonably regular" way. Then any series summable (λ, k) to s is also summable $(\phi(\lambda), k)$ to s. This theorem, due in its original form to Hardy, was later proved under less restrictive conditions on $\phi(x)$ by K. A. Hirst.

(B). Let $\phi(x)$ be a positive function increasing to infinity as $x \to \infty$, which is such that, for any δ (however small), $\phi(x) = O(x^{\delta})$, and which satisfies similar "regularity conditions." Suppose that a given series, while not necessarily summable (λ, k), is summable (λ, k') to s for some k', and that its (λ, k) mean is $o(F(x))$, where $F(x)$ is a certain function depending on $\phi(x)$. Then the series is summable $(\phi(\lambda), k)$ to s. This theorem is due to Zygmund.

These two theorems have usually been regarded as quite separate theorems, albeit belonging to the same field of ideas and proved by similar methods. An interesting feature of the treatment in this book is that the authors have unified the subject by giving a general theorem which includes both (A) and (B) as special cases. This is done at the cost of imposing conditions on $\phi(x)$ which are slightly more restrictive than those imposed by Hirst, but this difference is not of great importance. Unfortunately, there is an error in the argument which renders the proof invalid as it stands. (The line immediately above equation (2.59) does not follow from the previous line if $p<0$.) It would, however, be possible to construct a correct proof without any very extensive alterations to the text.

One other remark may be made. The authors assert that, under the conditions of (B), the method $(\phi(\lambda), k)$ is more powerful than (λ, k). While this is no doubt the case for any "reasonable" sequence $\{\lambda_n\}$, it is surely not *always* true. For if λ_n increases so rapidly that the conditions of the "high indices theorem" are satisfied not only by $\{\lambda_n\}$, but also by $\{\phi(\lambda_n)\}$, then (λ, k) and $(\phi(\lambda), k)$ are both equivalent to convergence.

Chapter III is concerned with the summability (λ, k) and (e^λ, k) of the Dirichlet series

$$\sum_{n=0}^{\infty} a_n e^{-\lambda_n s}.$$

It was, of course, inevitable that this chapter should contain a certain amount of material which is also to be found in Hardy and M. Riesz's Cambridge Tract on Dirichlet series. But much other material is also included. For example, a fair amount of space is devoted to results concerning absolute Riesz summability, the theory of which has been developed only since the publication of the Tract. Again, the problem of obtaining sufficient conditions for the summability of the series at a point lying actually on the abscissa of summability, which is only briefly mentioned in the Tract, is dealt with at some length.

The title of Chapter IV (*Applications to Fourier series*) is perhaps slightly misleading; one might expect it to refer to single Fourier series, whereas the chapter is throughout concerned with the "spherical" summability of multiple Fourier series. But, of course, any theorem on k-fold Fourier series which is valid for all k will yield results on single Fourier series on taking $k=1$, and a number of important results on single Fourier series are, in fact, included as special cases of the theorems of this chapter. The greater part of this chapter is concerned with the results of quite recent research, and I found it of much interest. Unfortunately, it is marred by an error which renders the proof of one of the theorems invalid as it stands. (Equation (4.54) does not follow from the line above, and is, in any case, clearly false, for its truth for arbitrarily small t would imply that $S(x)$ was continuous.)

Apart from the errors which have been mentioned, the book contains a number of slips of minor importance.

B. KUTTNER

Volume and integral. By W. W. Rogosinski. New York, Interscience, 1952. 10+160 pp. $1.75.

This little book is an exposition of the classical Lebesgue theory of Euclidean n-space, following the outer-inner measure approach to measure theory, and the ordinate set approach to integration theory. The book is intended as an introduction to modern abstract integration theory. More specifically, the author has in mind the incorporation of Lebesgue theory into the undergraduate honors programs of English universities. He has therefore chosen to present a detailed exposition of a restricted portion of the subject.

The first chapter takes up the necessary amount of Boolean algebra (with countable operations) and point set topology in Euclidean n-space. The third chapter develops measure theory for n-space by first defining measure for countable unions of "intervals" and then using these sets to define outer and inner measure for general sets. The fifth chapter presents the ordinate set approach to integration, wherein the integral of a non-negative function is defined as the "volume" under its graph in $(n+1)$-space. Included are the basic convergence theorems and the Fubini theorem, but not L^p theory. The sixth chapter takes up the theory of differentiation in one dimension, based on the Vitali covering theorem.

The most novel feature of the book is the inclusion, in chapters two and four, of an independent parallel development of the theory of content and the theory of Riemann integration based on the theory of content. The student is thus led through the historical evolution of the subject, and can better appreciate the tremendous stride taken by Lebesgue.

Proofs of theorems are given in detail. Scattered throughout the book is a collection of 35 exercises, indications of their proofs being collected at the ends of chapters. The book is carefully written, and should admirably serve its purpose as an undergraduate text.

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