Lagerungen in der Ebene, auf der Kugel und im Raum. By L. Fejes Tóth. (Grundlehren der mathematischen Wissenschaften, vol. 65.) Berlin, Springer, 1953. 10+197 pp. 24 DM.

This is probably the only book, in any language, devoted to the subject of "arrangements" such as packings and coverings. It is full of interesting results, many of them discovered by the author himself, and the rest collected from a great variety of sources. Yet it is not a collection of isolated theorems but develops the subject systematically. The exposition is clear, and relieved by frequent historical interludes (such as one drawing attention to Minkowski's enthusiastic remark: "Mich interessiert alles, was konvex ist!"). There are 124 beautiful figures, a five-page bibliography (including many works as recent as 1951 and 1952) and a useful index.

After a brief introduction to the theory of convex regions, the author gives a neat proof that, if two ellipsoids are polar reciprocals with respect to a unit sphere, their volumes, E and E', satisfy

$$EE' \geqq (4\pi/3)^2,$$

with equality only when the ellipsoids and sphere all have the same center. This is essentially a theorem of affine geometry: If two ellipsoids are polar reciprocals with respect to a third, the geometric mean of their volumes is greater than or equal to the volume of the third. Another affine theorem, this time in two dimensions, is that, if an n-gon contains an ellipse of area e and is contained in an ellipse of area E, then

$$e/E \leqq \cos^2 \pi/n.$$

This first chapter includes also a nicely illustrated account of the regular and Archimedean solids and of the analogous tessellations; e.g., (3, 3, 4, 3, 4) is the tessellation of triangles and squares in which no two squares share a side (so that each vertex is surrounded by two triangles, a square, another triangle, and another square, as the symbol indicates). The author might well have mentioned (on p. 19) that this particular tessellation is not anomalous like (3, 3, 3, 4, 4), but can be derived from the regular tessellation (4, 4, 4, 4) in the same way that the snub cube (3, 3, 3, 3, 4) is derived from the cube (4, 4, 4) or from the octahedron (3, 3, 3, 3) (cf. Coxeter, *Regular and semiregular polytopes*, I, Math. Zeit. vol. 46 (1940) pp. 380–407, especially p. 395).

Chapter II includes a good exposition of Blaschke's important concept of the *affine length* of a curve. It is proved that the affine circumference λ of an oval curve of area T satisfies

$$\lambda^3 \leqq 8\pi^2 T,$$

with equality only for an ellipse (whereas the ordinary circumference L satisfies

$$L^2 \geqq 4\pi T,$$

with equality only for a circle).

In Chapter III we come to one of the problems from which the book takes its name: the closest packing and thinnest covering of the plane with equal circles, or, as the author vividly explains: the most efficient distribution of trees in an orchard, and of oases in a desert. (It is a pleasant feature of the book that the exquisitely precise diagrams for these problems contain quite recognizable trees, and even an Arab riding a camel to the nearest oasis.) With any discrete set of points (such as the trees or oases), the author associates a decomposition of the plane into polygonal "cells" (elsewhere called "Dirichlet regions"). Each of the points lies in one cell, whose interior consists of all points that are nearer to this point than to any other one of the set. Given the cells, their inscribed circles form a packing and their circumscribed circles form a covering. The solution of the packing problem (p. 67) is that if a convex region of area T contains a set of at least two nonoverlapping congruent circles, the sum of their areas is less than $\pi T/2 \cdot 3^{1/2}$. The solution of the covering problem is that, if a convex region of area T is completely covered by a set of at least two congruent circles, the sum of their areas is greater than $2\pi T/3 \cdot 3^{1/2}$. It follows that the best positions, both for the trees and for the oases, are the centers of the cells of the regular tesselation of regular hexagons, (6, 6, 6). (A result that many of us could guess but few could prove!)

This chapter includes also some results on regions of various sizes. For instance, if L is the sum of the circumferences of n nonoverlapping circles in a convex hexagon of area S, then

$$L^2 \leqq 2\pi^2 nS/3^{1/2}.$$

(On p. 89 this expression is accidentally written as $\pi nS/3^{1/2}$.) The corresponding result for Λ, the sum of the affine circumferences of n nonoverlapping ovals, is

$$\Lambda^3 \leqq 72n^2 S.$$

The historical remarks include a reference to the independent discovery by Reifenberg, Bateman, and Erdös that 18 equal circles (and no more) can have at least one point in common with a given circle of the same size, while none of the circles has its center interior to another. Those authors described the system, but here for the first time we have an actual drawing (p. 96), which shows that the centers of the 18 circles are 18 vertices of the Archimedean tessellation (3, 4, 6, 4).

Chapter IV deals with arrangements of regions derived from one convex region by a group of translations. The density of the densest packing in such an arrangement is 2/3 when the region is a triangle, and greater otherwise; that of the thinnest covering is 3/2 when the region is a triangle, and less otherwise. (These results are ascribed to Fáry.) The state of affairs is interestingly different if the region is forced to have central symmetry. Then the density of the thinnest covering is $2\pi/3\cdot 3^{1/2}$ when the region is an ellipse, and less otherwise, while that of the densest packing is $\pi/2\cdot 3^{1/2}=0.9069\cdots$ when the region is an ellipse, but not always greater otherwise! It was shown independently by Reinhardt and Mahler that it has the smaller value

$$(9 - 4\cdot 2^{1/2} - \log 2)/(2\cdot 2^{1/2} - 1) = 0.9024\cdots$$

for a "smoothed octagon": a regular octagon whose corners are cut off by certain arcs of hyperbolas.

One of the most surprising results (p. 86) is that the maximum density of such a "lattice" packing of centrally symmetrical convex regions is as great as the density of any irregular packing of congruent regions of the same shape. For instance, a packing of congruent ellipses cannot be improved by allowing their major axes to lie in various directions.

Extremal properties of the regular polyhedra are considered in Chapter V. It is proved that the density of a packing of $n>2$ small circles on a sphere is

$$\leqq \frac{n}{2}\left(1 - \frac{1}{2}\csc\omega_n\right),$$

where

$$\omega_n = \frac{n\pi}{6(n-2)},$$

while that of a covering by $n>2$ small circles is

$$\geqq \frac{n}{2}\, n(1 - 3^{-1/2}\cot\omega_n).$$

In either case equality occurs only when the centers of the circles are the vertices of an equilateral triangle (inscribed in a great circle), a regular tetrahedron, on octahedron, or an icosahedron. If a polyhedron with n vertices or n faces contains a sphere of radius r and is contained in a sphere of radius R, then

$$R/r \geqq 3^{1/2} \tan \omega_n.$$

The corresponding affine theorem, about volumes of ellipsoids, is obtained by cubing both sides. For a polyhedron with k edges, we have the same inequalities with ω_n replaced by $k\pi/6(k-3)$.

A new proof is given for Michael Goldberg's form of the isoperimetric theorem: The surface F and volume V of a convex n-hedron satisfy

$$F^3/V^2 \geqq 54(n-2) \tan \omega_n(4 \sin^2 \omega_n - 1).$$

It is also proved that the sum of the edges of a convex polyhedron containing a sphere of diameter D satisfies

$$L > 10D.$$

If we are given that all the faces have the same area, "$>10D$" can be replaced by "$\geqq 12D$", with equality only for a cube of edge D.

The above results for small circles on a sphere may be interpreted as applying to circles in the elliptic plane, and suggest corresponding properties of the hyperbolic plane. In particular, the density of a packing of equal circles in the hyperbolic plane is always less than

$$3/\pi = 0.9549 \cdots .$$

(For the Euclidean plane, "$<3/\pi$" has to be replaced, as we have seen, by "$\leqq \pi/2\cdot 3^{1/2}$.")

The packing problem for n equal small circles on a sphere is taken up again in Chapter VI, where the diameter of the circles in the densest packing is denoted by a_n, so that

$$\cos a_n \geqq \frac{1}{2} (\cot^2 \omega_n - 1),$$

with equality when $n=3$, 4, 6 or 12. Values of a_n are found for $n \leqq 16$ in the manner of Schütte and van der Waerden (*Auf welcher Kugel haben* 5, 6, 7, 8 *oder* 9 *Punkte mit Mindestabstand Eins Platz*? Math. Ann. vol. 123 (1951) pp. 96–124). In particular, it is proved that $a_5 = a_6 = \pi/2$,

$$\cos a_7 = \frac{1}{2} (\cot^2 \omega_8 - 1), \quad \text{and} \quad \cos a_8 = (2\cdot 2^{1/2} - 1)/7,$$

the centers of the 8 circles being the vertices of the square antiprism (3, 3, 3, 4).

Finally, Chapter VII deals with arrangements of equal spheres in Euclidean space. If all the spheres are derived from one by a group of translations, the cell (Dirichlet region) for the densest packing is a rhombic dodecahedron, while the cell for the thinnest covering is a truncated octahedron (4, 6, 6). In other words, the centers in the densest lattice-packing form a face-centred cubic lattice, while the centers in the thinnest lattice-covering form a body-centered lattice. The latter result is hinted at on p. 189, but Bambah's proof was written just too late to be mentioned in the book.

For nonlattice arrangements, however, the problems present difficulties that have not yet been overcome. A solid sphere can be surrounded by twelve equal solid spheres, all touching it, in various ways, and it seems plausible that (as James Gregory believed) the twelve might be bunched together in such a way as to leave room for a thirteenth, still touching the original one. However, some very recent work of Schütte, Boerdijk, and van der Waerden shows this to be impossible; in fact, the distance from the center of the original sphere to the center of the thirteenth surrounding sphere seems to be at least

$$7/3\cdot 3^{1/2} = 1.347 \cdots$$

times the diameter.

The book contains many other interesting results and discussions that cannot be mentioned here for lack of space. Some are simple enough for high-school students to appreciate and yet sufficiently unfamiliar to delight a professional mathematician. Moreover, there are enough unsolved problems to provide material for research for many years to come.

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