BOOK REVIEWS

Calculus, a modern approach. By K. Menger. 2d ed. Illinois Institute of Technology Bookstore, 1953.

This book is the outcome of a long program of research by Professor Menger to simplify the ideas and notation of calculus. The most striking feature of it is the treatment of variables.

The classical variable, for which the notations of calculus were developed, was vaguely a generic symbol x which stands for the elements of a set X. This is akin to the algebraist's idea of an indeterminate. In the classical variable terminology, the symbol $f(x)$ represents a variable which is a function of the variable x. The Weierstrassian variable x was a symbol which stands for a "fixed" ("arbitrary") but unspecified element of a set X. Ordinarily a proposition involving a Weierstrassian variable is preceded by quantification. In terms of Weierstrassian variables the symbol $f(x)$ is regarded as representing the value of the function f at x.

The confusion between the symbol which represents the function and that which represents its evaluation remains a serious defect in the language of calculus to the present day, for although a writer may set out to employ a pure Weierstrassian point of view, and though all goes well with the basic topological theory, he gets into trouble with the machinery of calculus when he comes to speak of the function x^2+1, or $\sin(n\theta+\phi)$. The trouble grows deeper when he begins to cope with substitution theory, with multiple integrals and partial derivatives.

Menger retains the Weierstrassian variable with more or less standard notation but he makes a sharp break with the classical variable x whose domain is the set X, replacing it by the identity function I on X to X. The result is not only a conceptual clarification and simplification of the ideas of calculus, but, when this program is worked out in detail in an elementary text, there appear some illuminating perspectives on the rôles of the two languages. The Weierstrassian language works well for the function theoretic foundations, but once we are in possession of the properties of continuous functions on compact intervals, the calculus proper is better served by Menger's modern substitute for the classical variable language. Essentially what happens is that the introduction of the identity function in place of a variable carries the subject upstairs into the realm of linear operators in the algebra $C(x)$ of continuous functions.

This is the proper framework for calculus; that is, for the representational and calculational aspects of the theory.

In the opinion of the reviewer it would be a mistake to underestimate the importance of Menger's contribution by regarding it as merely a notational change. Moreover, in this pioneer work it is essential to the author's purposes to adopt a new system of symbols in order to emphasize the distinctions. However, for the purposes of this review it may be helpful to consider the notational effects of making the opposite choices to the ones made by Menger. Since a third meaning of the symbol x has always been the identity function; let us adopt this meaning for x and distinguish the other variable ideas by different devices. For the Weierstrass variable in the domain of x a time-honored device is to replace x by x_1 or x_2. A second device is to replace x by one of the earlier letters of the alphabet, as in $f(b)-f(a)$. A third device could be used for emphasis, square brackets, $f[a]$ for "f at a"; thus saving parentheses for composition as in $\sin(2x+1)$. Then $f(x)$ becomes identical with $f\colon X\to R$ and has the added virtue that $f(x)$ indicates more about the domain than f does. Then $\int_E f(x)dx$ means the integral of f over some set E in its domain, X, with respect to the identity function $x\colon X\to X$. And $D_x(2x^2+1)$ has a proper meaning which can be achieved only by involved circumlocution in terms of Weierstrass variables. Menger writes this $\mathcal{D}(2I^2+1)$.

Actually a second basic function is needed to make the scheme elegant if not to make it conceptually correct. This is the one-function $\mathit{1}\colon R\to R$ defined for every a in R by $\mathit{1}[a]=1$. With it, the polynomial ring $P[x, \mathit{1}]$ becomes algebraically isomorphic to the polynomial ring $P[x, 1]$ over R in which x is an indeterminate. But now $P[x, \mathit{1}]$ is a ring of polynomials *as functions* $p(x)\colon R\to R$. Menger uses the notations c and $\underline{c}$ for constant functions which is rather inconsistent with the idea of generating functions by ring operations in $C(x)$.

A second outstanding feature of this book is the author's painstaking approach to the mathematical representation of physical quantities. Here he virtually abandons his identity function as a variable in favor of a scheme of pairing the states of a physical observable with numbers in such a way that the "physical variables" which emerge are apparently old fashioned scalar variables. In view of the good start towards the representation of physical quantities by linear operators which is inherent in the author's theoretical treatment of variables and functions, and in view of the now well established physical doctrine, "to every physical observable there corresponds a linear operator," why did the author set up a separate idea of a physical variable?

The reviewer pondered this question at some length and offers the following conjecture. There are several closely related geometries in the Cartesian plane, the confusion of which can be as troublesome as the confusion of different concepts of variable, *especially in the mathematical representation of physical quantities*, or "dimensional analysis" in the physicist's sense. The principal geometry of the Cartesian plane for the purposes of measurement is that of the Cartesian product of the affine line and affine line $A_1 \times A_1$ in which independent transformations, $x' = ax + b$, $y' = cy + d$, $a \neq 0$, $c \neq 0$, are admitted. Physical laws must have statements invariant in this geometry. The subject of dimensional analysis lives in this geometry. However, if in apparently innocuous indulgence, the more familiar Euclidean plane geometry E_2 is adopted instead, the subject of dimensional analysis vanishes, leaving behind only ghosts whose goings on are to be recorded by fiat. The author is not explicit about his choice of geometry but the reviewer's conjecture is that his point of view about "physical variables" is that of a man who is thinking in E_2.

In the opinion of the reviewer a consistent operator approach to the representation of physical quantities in the elementary sense is still needed to prepare the way for this point of view which forces itself upon us of necessity in more advanced situations. The author of this book has laid the groundwork for it by introducing the identity function in lieu of the scalar variable, so that his linear operators $\mathfrak{D}$ and $\mathfrak{D}^{-1}$ work properly. The natural extension of this same idea should do much to simplify dimensional analysis.

Another interesting aspect[1] is that the variable used by Menger is much closer to the idea of a random variable than the conventional one, in fact, *is* a random variable with uniform distribution.

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Elements of number theory. By I. M. Vinogradov. Trans. from the 5th rev. ed. by S. Kravetz. New York, Dover, 1954. 8+227 pp. $1.75 paper; $3.00 cloth.

This elementary book was originally published in 1936 with a little under 100 pages and subsequent editions appeared every few years until the sixth in 1952 of almost double the size. This work, as most other Russian books, has not been easy to obtain in the past and its present appearance is gratefully noted. It is a very welcome addition to the books on number theory.

As has been observed by other reviewers, the main body of the

[1] Pointed out by R. N. Bradt.