conclude that $(y, e^{\delta/2}y)$ is a subinterval of $(x, \rho x)$ since one only knows that $x \leqq y \leqq \rho x$.

Because this book is well conceived and gives a good picture of the elementary results, it is unfortunate that the execution suffers from the defects noted above. For the more mature student who reads with a skeptical eye, these defects are not of major importance but one cannot recommend this work to the unwary.

LOWELL SCHOENFELD

Entire functions. By R. P. Boas, Jr. (Pure and Applied Mathematics, vol. 5.) New York, Academic Press, 1954. 12+276 pp. $6.00.

This is the first book which is devoted primarily to the theory of analytic functions of exponential type, and it gives a practically complete account of the subject. A function $f(z)$ is said to be of exponential type when it is regular in an unbounded set, often the whole plane, and satisfies an inequality $|f(z)| \leqq Ae^{c|z|}$ on the set. Besides the theory of these functions the book contains a large chapter on entire functions in general, a chapter on the minimum modulus of an entire function, and a chapter on applications of functions of exponential type. The author and his publisher have rendered a great service by making so much material available in a single volume of moderate size and price.

One may regard Pólya, Valiron, Paley and Wiener, and S. Bernstein as the founders of the present theory of functions of exponential type. An unusually large number of mathematicians contributed to its further development (the bibliography lists close to 400 papers!). Of these, the author has probably contributed to more aspects of the theory than anybody else. Moreover, because of his past and present association with Mathematical Reviews the author was in a unique position to become thoroughly familiar with the extensive literature on the subject, including the Russian literature. In addition to these qualifications the author possesses considerable skill as an expositor, and he has thus produced a book which is both extremely useful and very readable.

The author's job has not been easy. Faced with an overwhelming amount of material and striving to be complete he had the tremendous task of ordering a veritable chaos. In the opinion of the reviewer he has succeeded exceedingly well. A good example is furnished by Theorem 8.4.16 for whose proof Levinson (1940) needed 14 pages. In the present book the theorem is just one result in an integrated theory and its proof takes only a little more than half a page. In this and other places the author's method of dealing with a class of related theorems is noteworthy. He begins by proving a simple result, which gives him an opportunity to exhibit the essential ideas of the proofs. Next he either proves the most general result of its kind, or he states the best generalizations that are known without proof, but with reference to the literature. (No use is made of these starred theorems whose proof is omitted.) The above features should make the book very attractive both as an advanced text and as a handbook for nonspecialists who are looking for applicable results. For the student interested in research in the field the reviewer would have welcomed a list of unsolved problems which the author deems worthy of attention.

A description of the content of each chapter follows.

Chapter 1 (7 pages) deals briefly with some of the tools that are frequently used throughout the book: Jensen's and Carleman's formulas, Poisson's formula for a semi-circle, Phragmén-Lindelöf theory, and various kinds of density.

Chapter 2. General properties of entire functions of finite order (31 pages). The author calls an entire function $f(z)$ of growth (ρ, τ) when it is of order $\leqq \rho$ and of type $\leqq \tau$ when of order ρ. An entire function of exponential type τ is then defined as a function of growth $(1, \tau)$; similarly for non-entire functions. For entire functions a great many connections are established between growth and zeros—many more than in any other book. Here one finds Borel's proof of Hadamard's factorization theorem and Lindelöf's very useful theorems relating type and zeros, results of Valiron and the more recent work of Shah.

Chapter 3. The minimum modulus (16 pages). Since Borel compared $\log m(r)$, the minimum of $\log |f(z)|$ on $|z| = r$, with $-r^{\rho+\epsilon}$, considerable progress has been made. On the one hand one finds here the results which show that, for entire functions of small order, $\log m(r)$ is occasionally quite large—for example, as large as $r^{\rho-\epsilon}$, when $\rho < 1/2$ (Wiman). On the other hand there are the much more recent weaker inequalities which hold on large sets. Typical of the latter is a result of Chebotarëv and Meiman for entire functions $f(z)$ of growth (ρ, τ), $\tau < \infty$, which shows that $\log |f(z)|$ is practically everywhere greater than $-Ar^{\rho}$. Applications of these results are made in later chapters where the author has to divide by given entire functions. The chapter also deals with a large class of results in which $m(r)$ is compared with the maximum modulus $M(r)$. Hayman's recent work on the minimum modulus is briefly mentioned in chapter 2.

Chapter 4 (11 pages) deals with Valiron's theorem on entire functions $f(z)$ of order ρ, $0 < \rho < 1$, all of whose zeros are real and negative.

When $n(r)$ denotes the number of zeros of $f(z)$ in $|z| \leqq r$ the theorem says that the assertions $n(r) \sim cr^{\rho}$ $(c>0)$ and $\log f(r) \sim \pi c$ $(\csc \pi\rho) r^{\rho}$ are equivalent. Various generalizations are treated, among them one of the author in which $\log |f(-r)|$ also plays a role, and one due to Pfluger which allows a much more general distribution of zeros.

Chapter 5 (16 pages) roughly follows Pólya's paper (Math. Zeit. vol. 29) in its exposition of the properties of the Phragmén-Lindelöf indicator function

$$h(\theta) = \limsup r^{-1} \log |f(re^{i\theta})| \qquad (r \to \infty)$$

and the indicator diagram. One finds here in particular the two equivalent definitions of the conjugate indicator diagram D of the entire function $f(z) = \sum a_n z^n$ of exponential type: it is the bounded closed convex set whose supporting function is $h(-\theta)$, and also the smallest closed convex set outside which the Borel transform $F(z) = \sum n! a_n / z^{n+1}$ of $f(z)$ is regular. Pólya's representation

$$f(z) = (2\pi i)^{-1} \int_C F(w) e^{zw} dw$$

(where C contains D) is proved as a tool for later chapters, and so are Macintyre's extensions of these results to functions which are of exponential type in an angle.

Chapters 6, 7. Functions of exponential type, restricted on a line (51 pages). Among the three most important results are two representation theorems for functions $f(z)$ which are regular and of exponential type for $y \geqq 0$, and "restricted" along the real axis. Let $h(\pi/2) = c$. (i) When $f(x) \in L^2(-\infty, \infty)$ one has Paley and Wiener's representation

$$f(z) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-c}^{A} e^{izt} \phi(t) dt, \qquad \phi(t) \in L^2(-c, \infty),$$

which becomes particularly simple and useful in the case of an entire function of exponential type. (ii) When $\int_{-\infty}^{\infty} (1+x^2)^{-1} \log^+ |f(x)| dx < \infty$ one has Nevanlinna's representation

$$\log |f(z)| = \log |B(z)| + y\pi^{-1} \int_{-\infty}^{\infty} \frac{\log |f(t)|}{(t-x)^2 + y^2} dt + c'y,$$

where

$$B(z) = \prod \frac{1 - z/z_n}{1 - z/\bar{z}_n}$$

is the Blaschke product formed with the zeros z_n of $f(z)$ in $y>0$. ((ii) is actually established under weaker conditions.) The author next proves the theorem of Ahlfors and Heins (Ann. of Math. vol. 50) which gives a quite accurate description of the asymptotic behavior of $\log |B(z)|$. (iii) It follows that under the assumptions of (ii) the quantity $r^{-1} \log |f(re^{i\theta})|$, in the terminology of the author, tends effectively to $c' \sin \theta$ in $0<\theta<\pi$ when $r \to \infty$. Thus in particular $c'=c$. A large variety of applications of (iii) are made.

Chapter 8 (19 pages) connects growth and distribution of zeros in the case of entire functions of exponential type whose zeros z_n are close to the real axis in the sense that $\sum |\operatorname{Im}(1/z_n)|$ converges. The principal result of the chapter grew out of work by Paley and Wiener, with Pfluger its main architect (Comment. Math. Helv. vol. 16), and Noble and the author adding a finishing touch. Let $f(z)$ satisfy the above hypotheses, and let $f(0)=1$. Then the following three conditions are equivalent. (i) $\lim n(r)/r$ exists; (ii) $\lim \int_{-r}^{r} x^{-2} \log |f(x)| dx$ exists; (iii) $\lim r^{-1} \log |f(ir)|$ exists if a set of finite logarithmic length is excluded. Various related results of Pfluger, Levinson and Cartwright are also proved.

Chapter 9. Uniqueness theorems (26 pages). Most theorems in this chapter state that a function with certain growth properties must vanish identically if it or a certain transform have too many zeros. (i) A function regular for $x \geqq 0$ and of exponential type less than that of $\sin \pi z$ which vanishes on the positive integers is identically zero (Carlson). Many generalizations are given in which the growth conditions are weakened and/or the sequence $\{n\}$ is replaced by a more general sequence $\{\lambda_n\}$. Of these, Fuchs's theorem is especially interesting because it is in a certain sense a best possible result. (ii) There is a similar class of results for entire functions vanishing on or near the integers, with somewhat weaker growth conditions (Pólya, Valiron and others). (iii) When $f(z)$ is entire and of exponential type less than 1, and $f^{(n)}(n)=0$, $n=0, 1, \cdots$, then $f(z) \equiv 0$. Results of the kind (i) and (ii) are derived from the growth properties of simple specific functions vanishing at the points n or λ_n, while (iii) is obtained as a special case of a general expansion theorem due to Buck.

Chapter 10. Growth theorems (28 pages). In this chapter the typical theorem asserts that a function of exponential type which is of certain growth on a sequence of points on or near some line has certain growth all along the line. The chief tool is a suitable interpolation series on the given sequence of points which should not differ too much from the function along the given line. Let $f(z)$ be regular and of exponential type in $|\arg z| \leqq \alpha \leqq \pi/2$, and let $h(\theta) \leqq a|\cos \theta|$

$+b \sin \theta$, $b<\pi$. Then the main theorems are as follows. (i) If $\{\lambda_n\}$ is an increasing sequence of positive numbers such that $\lambda_{n+1}-\lambda_n \geqq \delta > 0$, $|\lambda_n - n| \leqq L$, and $\{f(\lambda_n)\}$ is bounded, then $f(x)$ is bounded for $x>0$ ($\lambda_n = n$: Cartwright; general case: Duffin and Schaeffer). (ii) If $\{\lambda_n\}$ is a (complex) sequence such that $n/\lambda_n \to 1$ and $|\lambda_m - \lambda_n| \geqq \delta |m-n|$, $\delta > 0$, then lim sup $|\lambda_n|^{-1} \log |f(\lambda_n)| = h(0)$ ($\lambda_n = n$: Pólya; general case: V. Bernstein).

Chapter 11. Operators and their extremal properties (27 pages). One must be grateful indeed that the author has included this very interesting material. Until now a considerable portion of it could be found only in the Russian literature. Let L be an operator which carries every entire function of exponential type into another such function. Question: does the rate of growth of $L[f(z)]$ along a line follow from that of $f(z)$ on a certain set of points on the line? The main interest is in obtaining exact bounds. We cite two theorems due to S. Bernstein; $f(z)$ is supposed to be of exponential type τ and bounded on the real axis. (i) If $|f(x)| \leqq M$ for all real x, then $|f'(x)| \leqq \tau M$. (ii) $|f(x+\pi/2\tau)+f(x-\pi/2\tau)| \leqq (8/\pi)$ sup $|f(n\pi/\tau)|$. Among the many refinements and related results given in the chapter are inequalities due to Szegö, Schaeffer, and the author. Two methods of proof are exhibited. The first, developed by Civin and also by Krein, applies to operators L which carry functions $f(z) = \int_{-\tau}^{\tau} e^{izt} d\alpha(t)$ into $L[f(z)] = \int_{-\tau}^{\tau} \lambda(t) e^{izt} d\alpha(t)$ with a fixed $\lambda(t)$. The second method of proof depends on the use of an interpolation series for $f(z)$ due to Valiron. The chapter closes with a discussion of Levin's B-operators, which enable one to prove results of the form "$|f(x)| \leqq |\omega(x)|$ implies $|L[f(x)]| \leqq |L[\omega(x)]|$."

Chapter 12. Applications (19 pages). These include results on the asymptotic behavior of moments and the completeness of sets $\{t^{\lambda_n}\}$ and $\{\exp(i\lambda_n x)\}$. Uniqueness theorems for Fourier series are proved, and theorems on the behavior of a function represented by a power series on or near the circle of convergence. Finally there are theorems on expansions of analytic functions in series of polynomials, on differential equations of infinite order and on approximation on the real axis by entire functions of exponential type.

A bibliography of 17 pages mentions the papers which are referred to in the book. Only one or two of these papers are prior to 1920, many are as recent as 1953. The book closes with an unusually complete index of 8 pages. (Surprisingly, the index even shows the extent of Bourbaki's contributions to the subject matter of the book.)

Several years before he wrote this book the author confided to the reviewer that on the average every page of mathematics he had seen contained a minor error, and that he had never seen 50 pages of mathematical work without a serious mistake. The author now informs the reviewer that a mimeographed list of errata for this book can be obtained by writing to him.

J. KOREVAAR

Probability theory. Foundations. Random sequences. By M. Loève. Toronto, Van Nostrand, 1955. 15+515 pp. $12.00.

The mathematical theory of probability has been well established in its modern form for about twenty years, but Loève's is the first (non-elementary) textbook on the subject. That is to say, there has been no textbook which, like a textbook covering any other advanced mathematical subject, gives the fundamental definitions, basic theorems, and enough further development to lead the reader into the really advanced literature. Feller's *Probability theory and its applications* (1950) is superlative as far as it goes (only through discrete probabilities), but the promised further volumes are still keeping company with the other unborn descendants of first volumes, an illustrious group. The reviewer's *Stochastic processes* (1953) has been the closest approach to a probability textbook, but, as its title indicates, this book was neither written as nor intended to serve as a general textbook, and its choice of topics and emphasis were dictated by its title. Authors have been willing to write specialized probability books, of which there have been many, besides the two just mentioned, by Bartlett, Blanc-Lapierre and Fortet, Fortet, Gnedenko and Kolmogorov, Ito, Lévy, but the drudgery involved in writing a systematic and complete text has not been attractive. Thus, even if Loève's book were not as successful as it is, he would still deserve the thanks and respect of the mathematical community for writing it.

The mathematical theory of probability is now a branch of measure theory, with certain specializations and emphasis derived from the applications and the historical background. As the historical conditioning loses its significance for newer generations of mathematicians, the place of probability theory in measure theory becomes more and more difficult to describe. One slightly frivolous description, which, however, is about as accurate a description as can be given, is that probability is the one branch of measure theory, and in fact the one mathematical discipline, in which measurable functions as such are considered in detail, and their integrals evaluated. (The fact that the integration of smooth functions on intervals can be considered as that of measurable functions is of course discounted here.) In fact,