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W. T. Martin

Part II. Complex manifolds[1]

The notion of a complex manifold is a natural outgrowth of that of a differentiable manifold. Its importance lies to a large extent in the fact that it includes as special cases the complex algebraic varieties and the Riemann surfaces and furnishes the geometrical basis for functions of several complex variables. Its development has led to clarifications of classical algebraic geometry and to new results and problems. Two notions from algebraic topology have so far played an essential rôle: sheaves (faisceaux) and fiber bundles. But the deeper problems on complex manifolds are not entirely topological.

1. **Topology of complex manifolds.** From the point of view of topology a fundamental problem would be to characterize the orientable manifolds of even dimension $2n$ which can be given a complex structure. But this is too difficult and, at least at the present moment, one

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should be satisfied with necessary conditions. An immediate necessary condition is obtained by the consideration of the tangent bundle.[2] For the existence of a complex structure implies that the tangent bundle whose structural group is the general linear group $GL(2n, R)$ in $2n$ real variables is equivalent to a bundle whose structural group is the general linear group $GL(n, C)$ in n complex variables, considered as a subgroup of $GL(2n, R)$. A manifold of even dimension $2n$ with the latter property is called almost complex. Various necessary conditions are known for a manifold to be almost complex.

Among such necessary conditions the most effective ones are expressed in terms of the characteristic classes of the manifold, if we make the further assumption that the latter is compact. These are the Stiefel-Whitney classes $W^i \in H^i(M, Z_2)$,[3] $1 \leqq i \leqq 2n$, and the Pontrjagin classes $p_k \in H^{4k}(M, Z)$, $1 \leqq k \leqq [n/2]$ [49]. If M is almost complex, its almost complex structure defines the Chern classes $c_k \in H^{2k}(M, Z)$, $1 \leqq k \leqq n$, [49]. The following relations between the characteristic classes give necessary conditions for a manifold to be almost complex [49]:

$$W^i = 0, \qquad i \text{ odd}, \tag{1}$$

$$\sum_{i=0}^{[n/2]} (-1)^i p_i = \sum_{i=0}^{n} (-1)^i c_i \sum_{i=0}^{n} c_i. \tag{2}$$

Further necessary conditions are obtained by considering cohomology operations on the characteristic classes, in particular, the Steenrod squaring and reduced power operations.

These conditions suffice to give the theorem that among the even-dimensional spheres only S^2 and S^6 are almost complex. In fact, the absence of an almost complex structure on S^{4k} was derived by Wu as a consequence of (2) and the facts: (1) $p_k = 0$; (2) $c_{2k} \cdot M = 2$. By using the formulas expressing $\wp_p^i c_j$ as a polynomial of $c_1, \cdots, c_n$ ($\wp_p^i$ is a Steenrod reduced power operation), Borel and Serre proved that S^{2n} is not almost complex for $n \geqq 4$ [6].

The question whether an almost complex manifold can be given a complex structure remains unanswered. In particular, the existence

[2] For basic notions on fiber bundles we refer the reader to Steenrod [42].

[3] We shall be using the following notations, now customary: Z denotes the ring of integers, Z_p (p prime) the finite field of p elements, R the real field, and C the complex field. If X is a topological space and G an abelian group, $H^r(X, G)$ (resp. $H_r(X, G)$) denotes the r-dimensional cohomology (homology) group with coefficient group G. If G is a ring and $\gamma \in H^r(X, G)$, $c \in H_r(X, G)$, then $\gamma \cdot c \in G$ denotes the pairing of the two groups into G. When M is an oriented manifold, the same notation will be used to denote its fundamental homology class.

or nonexistence of a complex structure on S^6 is still one of the urgent unsolved problems on complex manifolds. The difficulty lies in the fact that there is at present no method to find topological implications of the existence of a complex structure, which is not already true for the existence of the underlying almost complex structure. On the other hand, for a given almost complex structure, necessary conditions are known in order that it defines a complex structure (Ehresmann-Libermann-Eckmann-Frölicher [**16**; **17**]). These conditions are sufficient, if the almost complex structure is analytic.

Two complex structures on a manifold M are called inequivalent, if there exists no homeomorphism of M onto itself which transforms one complex structure into the other. Hirzebruch proved that the manifold $S^2 \times S^2$ has an infinite number of inequivalent complex structures [**20**]. In his study of homogeneous complex manifolds (cf. §6), Wang [**47**] gave examples of manifolds, among which are products of two spheres of odd dimensions >1, which have noncountably many inequivalent complex structures. However, it is undecided whether the complex projective plane has a complex structure inequivalent to its natural one.

As a counterpart of the birational transformations in algebraic geometry, Hopf introduced the modification or the σ-process [**26**; **44**] (cf. also the earlier work of Behnke-Stein [**3**]). Geometrically the process can be pictured as "blowing up" a submanifold which is complex-analytically imbedded in a complex manifold. It allows the construction of new complex manifolds from given ones. Various questions concerning the effect of this process on the invariants of complex manifolds remain to be studied.

2. **Complex analytic bundles.** Let Y be a complex manifold acted on by a complex Lie group G of complex analytic homeomorphisms. To define a complex analytic bundle over a complex manifold M with fiber Y we take a covering $\{U_i\}$ of M by coordinate neighborhoods. The bundle over U_i is homeomorphic to $U_i \times Y$ and its points have the local coordinates (z, y_i), $z \in U_i$, $y_i \in Y$, such that, if $z \in U_i \cap U_j$, the local coordinates (z, y_i) and (z, y_j) of the same point satisfy the relation $y_i = g_{ij}(z) y_j$, where g_{ij} (to be called the transition functions) defines a complex analytic mapping of $U_i \cap U_j$ into G. Perhaps the simplest case is when Y is a complex vector space of q dimensions and $G = GL(q, C)$. The bundle is then called a complex vector bundle. It is to be observed that it is the group G and the transition functions $\{g_{ij}(z)\}$, and not the fiber Y, which play a basic role in the properties of the bundle. With a notion of equivalence introduced in a natural way, a fundamental problem would be to determine all classes of

bundles over M with a given group G. So far little progress has been made toward this "classification problem." In the case of complex line bundles, that is, complex vector bundles with $q=1$ (cf. §3), the abelian character of the group $GL(1, C)$ makes it possible to introduce a group operation in the set of complex line bundles and thus allows a complete enumeration of the complex line bundles over a complex manifold. Another contribution to this problem was recently made by Grothendieck, who classified the complex vector bundles over the complex projective line (not yet published).

The topological theory of fiber bundles furnishes two tools which are of importance: the characteristic classes and the universal bundle theorem. The former are defined in terms of the underlying topological bundle of the analytic bundle. In spite of this there are problems concerning them which are not entirely of topological nature. For instance, it has been proved that, for an algebraic variety, the dual homology classes of the characteristic classes of the tangent bundle contain representative cycles which are algebraic [**11**]. This theorem can be proved by using the homology theory of fiber bundles, a method which will undoubtedly find further applications in the study of analytic bundles. On the other hand, the important homotopy methods in topological fiber bundles seem to have too much disregard for the complex structure and have so far not been found useful.

The Grassmann manifold $G(q, N)$ of all q-dimensional linear spaces through a point of a complex Euclidean space of dimension $q+N$ is, in an obvious way, the base space of a bundle of complex vector spaces of dimension q. The Grassmann manifold itself can be identified with the manifold of all $(q-1)$-dimensional linear spaces in a complex projective space of dimension $q+N-1$ and is a complex algebraic variety. As a result our bundle is an analytic bundle. A complex analytic mapping of a complex manifold M into $G(q, N)$ gives rise to a complex analytic vector bundle over M. In contrast to the topological case, it is not true that every complex vector bundle over M can be defined in this way, even if N is allowed to be sufficiently large.

If the base manifold M is an algebraic variety, a related theorem was proved by Nakano and Serre [**37**]: Let W be an analytic bundle of complex vector spaces of dimension q, with the transition functions $\{g_{ij}(z)\}$, $z \in U_i \cap U_j$, relative to a covering $\{U_i\}$ of M, $g_{ij}(z)$ being $q \times q$ nonsingular matrices. Let E be the complex line bundle defined by a generic hyperplane section of M, whose transition functions relative to the same covering of M are $\{f_{ij}(z)\}$. Then, for a sufficiently large positive integer m, the bundle $W(-E)^m$, with the transition functions $\{g_{ij}(z)f_{ij}^{-m}(z)\}$, is equivalent to one induced by a complex analytic mapping of M into a Grassmann manifold $G(q, N)$.

Two other groups appear prominently as the structure groups of analytic bundles: the nonhomogeneous complex linear group $G'(q)$ in q variables and the linear fractional group $K(q)$ in q variables, considered to be acting on the complex affine space E_q and the complex projective space P_q respectively. The corresponding bundles are called the affine and projective bundles. Projective bundles are of importance in algebraic geometry. These bundles are related to the vector bundles. In fact, the group $T(q)$ of translations is a normal subgroup of $G'(q)$ and $G'(q)/T(q)$ is isomorphic to $GL(q)$. Similarly, the group S of all scalar matrices λI ($\lambda \neq 0$, I = identity matrix) is a normal subgroup of $GL(q)$, and $GL(q)/S$ is isomorphic to $K(q-1)$. By taking projections of the transition functions into quotient groups, we get a vector bundle from an affine bundle and a projective bundle of one less dimension from a vector bundle. These bundles shall be called the derived bundles. A necessary condition for two affine bundles (or two vector bundles) to be equivalent is that their derived bundles are equivalent.

It turns out that the property for a projective bundle to be the derived bundle of a vector bundle of one more dimension is equivalent to another important property. The Grassmann manifold $G(q, N)$ is the base space of a bundle of projective spaces of dimension $q-1$. We say that a projective bundle is regular if it can be induced by a complex analytic mapping of M into $G(q, N)$. It follows from the theorem of Nakano and Serre that a projective bundle is regular if and only if it is the derived bundle of a vector bundle of one higher dimension.

Because of our present limited knowledge of vector bundles the classification of affine and projective bundles has to be restricted to line bundles ($q=1$). The first invariant of an affine line bundle is the derived complex line bundle. Over a compact complex manifold M consider all the affine line bundles which have the same derived complex line bundle F, but are inequivalent to it.[4] These affine line bundles can be set in a natural way into a one-one correspondence

[4] Relative to a covering $\{U_i\}$ of M let (z, y_i), $z \in U_i$, $y_i \in A$ (=affine line) be the local coordinates in the affine line bundle. Then, in $U_i \cap U_j$, the transition of coordinates is given by $y_i = a_{ij}(z)y_j + b_{ij}(z)$, where $a_{ij}(z)$, $b_{ij}(z)$ are holomorphic functions in $U_i \cap U_j$. The derived line bundle has by definition the transition functions $\{a_{ij}(z), z \in U_i \cap U_j\}$. The affine line bundle is said to be inequivalent to its derived complex line bundle, if it is not possible to make all $b_{ij}(z) = 0$ by an "analytic change of coordinates."

with the complex projective space derived from the vector space $H^1(M, \Omega(F))$, where $\Omega(F)$ is the sheaf of germs of holomorphic cross-sections of F [**27**; **36**; **48**]. If M is an algebraic variety, F is defined by a divisor class[5] D and $\Omega(F)=\Omega(D)$ is the sheaf of germs of meromorphic functions $> -D$. The dimension $i=i(D)=\dim H^1(M, \Omega(D))$ is then known as the index of specialty of D, if $\dim M=1$, and as the superabundance of D, if $\dim M=2$.

The study of a projective line bundle begins with the question whether it is equivalent to an affine line bundle. A necessary and sufficient condition for this is that the projective line bundle has an analytic cross-section. If this is the case, the projective bundle is regular. The converse is true if the base manifold is an algebraic curve A projective line bundle over an algebraic curve is always regular (unpublished result of Kodaira, Serre). The classification of projective line bundles over an algebraic curve reduces to finding conditions that the reduced affine line bundles are projectively equivalent. This problem was solved by Atiyah [**2**]. Atiyah is able to apply his theory to ruled surfaces with a high degree of success. Many classical results become easily accessible, and he is able to add new ones.

3. **Sheaves (faisceaux)**[6]. The cohomology groups of a manifold with a coefficient sheaf furnish the algebraic tool to formulate globally the properties of its local structure [**9**; **35**; **41**]. Its usefulness is based on the fact that the 0-dimensional cohomology group $H^0(M, \mathfrak{f})$ ($\mathfrak{f}$=coefficient sheaf) has a simple interpretation: It is the group of all cross sections $\Gamma(M, \mathfrak{f})$. But the consideration of the high-dimensional cohomology groups is important, because of the following fundamental property: Let $\mathfrak{g}$ be a subsheaf of $\mathfrak{f}$, and $\mathfrak{f}/\mathfrak{g}$ be the quotient sheaf. Then the cohomology groups are related by an exact sequence of homomorphisms

$$\begin{aligned} &0 \to H^0(M, \mathfrak{g}) \xrightarrow{i^0} H^0(M, \mathfrak{f}) \xrightarrow{j^0} H^0(M, \mathfrak{f}/\mathfrak{g}) \xrightarrow{\delta^0} H^1(M, \mathfrak{g}) \to \cdots \\ &\to H^q(M, \mathfrak{g}) \xrightarrow{i^q} H^q(M, \mathfrak{f}) \xrightarrow{j^q} H^q(M, \mathfrak{f}/\mathfrak{g}) \xrightarrow{\delta^q} H^{q+1}(M, \mathfrak{g}) \to \cdots. \end{aligned} \tag{3}$$

As an example we consider the case that $\mathfrak{f}=\mu$ is the sheaf of germs of meromorphic functions and $\mathfrak{g}=\Omega$ is the subsheaf of germs of holomorphic functions. A section of the quotient sheaf μ/Ω is a system of principal parts. The classical additive Cousin problem consists in deciding whether such a system of principal parts is that of a global meromorphic function. In our terminology the problem is that of

[5] This is essentially a part of the theorem of Lefschetz, cf. §3.

[6] For a more complete discussion cf. Part III.

characterizing the image $j^0H^0(M, \mu)$ in $H^0(M, \mu/\Omega)$, or, since the sequence is exact, the kernel of δ^0 in $H^0(M, \mu/\Omega)$. It follows that the additive Cousin problem always has a solution if $H^1(M, \Omega)=0$.

The most important cohomology groups of a complex manifold M with a coefficient sheaf are the groups $H^q(M, \Omega^p)$, where Ω^p is the sheaf of germs of holomorphic differential forms of type $(p, 0)$. Cartan-Serre and Kodaira [**10**; **28**] proved that the dimension $h^{p,q}$ of $H^q(M, \Omega^p)$ is finite if M is compact. For a Kähler manifold we have $h^{p,q}=h^{q,p}$. In general this relation is not true. Little is known about the combinations of $h^{p,q}$ which give topological invariants of M. For a Kähler manifold $\sum_{p+q=r} h^{p,q}$ is equal to the r-dimensional Betti number.

There already exist many applications of sheaves to the study of complex manifolds and classical algebraic geometry. Among them we mention the works of Kodaira-Spencer on the identification of different definitions of the arithmetic genera of algebraic varieties, on the lemma of Enriques-Severi-Zariski, the characteristic deficiency, etc. [**31**; **32**; **33**]. Hodge and Atiyah [**24**] applied sheaves to generalize a theorem of Picard-Lefschetz by proving that the maximum number of independent two-forms of the second kind on an algebraic variety is equal to $R_2-\rho$, where R_2 is the second Betti number and ρ is the Picard number.

As an illustration let us dwell a little bit more on the work of Kodaira-Spencer on the classification of complex line bundles [**32**]. All the complex line bundles over a complex manifold form an abelian group $\mathfrak{F}$. It contains as a subgroup the divisor-class group (because a divisor defines a line bundle in an obvious way). The study of the group of complex line bundles is based on the exact sequence of sheaves:

$$0 \to Z \xrightarrow{j} \Omega \xrightarrow{e} \Omega^* \to 0, \tag{4}$$

where Ω^* is the sheaf of germs of nonzero holomorphic functions, j is the inclusion mapping, and e is defined by $ef(z)=\exp\,(2\pi(-1)^{1/2}f(z))$. This exact sequence of sheaves gives rise to the following exact sequence of cohomology groups:

$$0 \to H^1(M, Z) \xrightarrow{j^*} H^1(M, \Omega) \xrightarrow{e^*} H^1(M, \Omega^*) \xrightarrow{\delta} H^2(M, Z) \to \cdots$$

It is easy to see that $\mathfrak{F}$ is isomorphic to $H^1(M, \Omega^*)$. If we identify these two groups, δ maps a complex line bundle into its characteristic class. The image $\delta H^1(M, \Omega^*)$ is the subgroup $H^2_{(1,1)}(M, Z)$ of $H^2(M, Z)$ consisting of all cohomology classes with integer coefficients which can be represented by a real closed form of type (1, 1). In case M is a Kähler manifold, this last condition is equivalent to saying that the harmonic part of the class is of type (1, 1). The kernel of δ or the image of e^* consists of those complex line bundles which are topologically product bundles. It is a complex abelian Lie group and is called the Picard variety $\wp$ of M. If M is a compact Kähler manifold, its Picard variety is a complex torus of dimension $R_1/2$, where R_1 is the first Betti number of M.

The quotient group $\mathfrak{F}/\wp$ which, according to the above discussion, is isomorphic to $H^2_{(1,1)}(M, Z)$ (or to the subgroup $H^{(1,1)}_{2n-2}(M, Z)$ of $H_{2n-2}(M, Z)$ by duality in M), has another important interpretation when M is a compact algebraic variety. In this case it is isomorphic to G/G_a, where G is the group of all divisors on M and G_a the subgroup of all divisors which are homologous to zero with integer coefficients. The resulting isomorphism between G/G_a and $H^{(1,1)}_{2n-2}(M, Z)$ is known as Lefschetz's Theorem. It can also be stated as a criterion for a $(2n-2)$-cycle to be algebraic (criterion of Lefschetz-Hodge, $n = \dim M$): An integral $(2n-2)$-cycle on an algebraic variety of dimension n is homologous to a divisor if and only if its harmonic part is of type (1, 1). It is a conjecture that this theorem has a generalization to $(2n-2q)$-cycles, to the effect that an integral $(2n-2q)$-cycle is homologous to an algebraic cycle if and only if its harmonic part is of type (q, q).

The sheaf Ω^p has a natural and important generalization $\Omega^p(F)$, which is the sheaf of germs of holomorphic p-forms in a complex manifold M with values in a line bundle F. The corresponding cohomology groups $H^q(M, \Omega^p(F))$ are again finite-dimensional if M is compact. Among these cohomology groups we have the following duality theorem of Serre [**40**]:

$$H^q(M, \Omega^p(F)) = H^{n-q}(M, \Omega^{n-p}(-F)). \tag{5}$$

In this isomorphism $-F$ denotes the line bundle whose transition functions are the reciprocals of those of F. Actually Serre proved his theorem for the more general case of vector bundles.

It is important in applications to find sufficient conditions for certain cohomology groups $H^q(M, \mathfrak{f})$ to vanish. For Stein manifolds there are the famous theorems A and B of Cartan and Serre (cf. §4). Serre found the analogues of these theorems in the cases of the complex projective space and of algebraic varieties in projective space. In the case of an algebraic variety M he showed that, for sufficiently large m, $H^q(M, \mathfrak{f}\otimes_\Omega(mE))=0$, $q\geqq 1$, where $\mathfrak{f}$ is a coherent analytic sheaf, E is the divisor defined by a hyperplane section, and Ω is the sheaf of germs of holomorphic functions in M.

While these theorems are adequate for many applications, Kodaira, by adopting a differential-geometric method originated from Bochner, obtained other sufficient conditions for the vanishing of the groups $H^q(M, \Omega^p(F))$, $q \geqq 1$, where F is a complex line bundle. We say that F is ample, if its characteristic class contains a representative closed quadratic differential form of type $(1, 1)$ whose corresponding Hermitian differential form is positive definite. Similarly, one defines the notion for a complex line bundle to be sufficiently ample. Kodaira proved that $H^q(M, \Omega^0(F))=0$, $q \geqq 1$, if $F-K$ is ample, K being the canonical bundle of M [**29**]. Moreover, if F is sufficiently ample, then $H^q(M, \Omega^p(F))=0$. On the other hand, Akizuki and Nakano proved that $H^q(M, \Omega^p(F))=0$, $p+q \leqq n-1$, if $F-K$ is ample [**1**]. These results are generalized by Spencer to vector bundles [**35**].

The interest in Kähler manifolds lies in the fact that many theorems on algebraic varieties are valid for compact Kähler manifolds. It is natural to ascertain the scope of this notion. The complex torus which does not satisfy the Riemann conditions gives an example of a nonalgebraic Kähler manifold. An important theorem of Kodaira states that, if a compact Kähler manifold is of restricted type, that is, if its fundamental two-form has integral periods over integral cycles, it is an algebraic variety [**30**]. The conjecture is not true that every compact Kähler manifold can be transformed by monoidal transformations into an analytic bundle of complex tori over an algebraic variety (A. Blanchard [**4**]).

4. **Stein manifolds.** The Stein manifolds (of dimension >0) are noncompact complex manifolds which generalize the domains of holomorphy and which possess a sufficiently large number of holomorphic functions. Precisely speaking, a Stein manifold M is a complex manifold with a countable base, satisfying the following conditions:

(1) To any two points p, $q \in M$, $p \neq q$, there exists a holomorphic function f in M, such that $f(p) \neq f(q)$;

(2) To every point $p \in M$ there exist n functions, holomorphic in M, which form a local coordinate system at p;

(3) M is holomorphically convex.

The following fundamental theorem accounts for most of the properties of Stein manifolds:

A complex manifold M is a Stein manifold if and only if the following two properties hold:

(A) For any coherent sheaf $\mathfrak{f}$ over M, the module of global cross-sections of $\mathfrak{f}$ over M generates at every point $p \in M$ the module of the local cross-sections $\mathfrak{f}_p$.

(B) For any coherent sheaf $\mathfrak{f}$ over M, $H^q(M, \mathfrak{f})=0$, $q \geqq 1$.

This theorem is due to Oka and Cartan [**9**; **39**]. The proof of the direct part, that is, that a Stein manifold has the properties (A) and (B), is difficult. The theorem has many consequences of which we mention the following:

(1) The additive Cousin problem (cf. §3) always has a solution on a Stein manifold;

(2) The second Cousin problem, the problem whether a given divisor is the divisor of a meromorphic function, has a solution on a Stein manifold if $H^2(M, Z)=0$;

(3) Every meromorphic function on a Stein manifold is the quotient of two holomorphic functions.

A first topological implication of a Stein manifold (of complex dimension n, and hence of real dimension $2n$) is that its p-dimensional homology groups $H_p(M, Z)$ with integer coefficients are, for $p>n$, torsion groups. It is not known whether they are all zero.

Another unsolved problem is to characterize the open submanifolds of a Stein manifold which are again Stein manifolds. It is also not known whether a covering manifold of a Stein manifold is a Stein manifold.

5. **Riemann-Roch theorem.** It has been known that the Riemann-Roch theorem can be formulated as a relation between the dimensions of certain cohomology groups with coefficient sheaves and the characteristic classes. Its exact formulation and proof were recently achieved by Hirzebruch [**22**; **51**]: Let M be an algebraic variety of dimension n, and W an analytic vector bundle over M, with fiber E_q and structural group $GL(q, C)$. Let c_i, $1 \leqq i \leqq n$, be the Chern classes of the tangent bundle of M, and d_j, $1 \leqq j \leqq q$, be the Chern classes of the bundle W. Denote by $\Omega(W)$ the sheaf of germs of holomorphic cross sections of W over M, and put

$$\chi(M, W) = \sum_{i=0}^{n} (-1)^i \dim H^i(M, \Omega(W)). \tag{6}$$

Introduce formally the quantities γ_i, $1 \leqq i \leqq n$, δ_j, $1 \leqq j \leqq q$, by the relations

$$\begin{aligned} 1 + \sum_{i=1}^{n} c_i x^i &= \prod_{i=1}^{n} (1 + \gamma_i x^i), \\ 1 + \sum_{j=1}^{q} d_j x^j &= \prod_{j=1}^{q} (1 + \delta_j x^j). \end{aligned} \tag{7}$$

Then the function

$$P(c_1, \cdots, c_n; d_1, \cdots, d_q) = (e^{\delta_1} + \cdots + e^{\delta_q}) \prod_{i=1}^{n} \frac{-\gamma_i}{e^{-\gamma_i} - 1} \tag{8}$$

is symmetric in γ_i and δ_j, and can be expressed as a power series in c_i, d_j. It can therefore be considered as a rational cohomology class of M. Following Hirzebruch, we put the symbol κ_{2n} before it to denote its value over the fundamental homology class of M. Then the Riemann-Roch-Hirzebruch theorem can be given as the formula

$$\begin{aligned} \chi(M, W) &= \kappa_{2n}\left[(e^{\delta_1} + \cdots + e^{\delta_q}) \prod_{i=1}^{n} \frac{-\gamma_i}{e^{-\gamma_i} - 1}\right] \\ &= \kappa_{2n}\left[e^{c_1/2}(e^{\delta_1} + \cdots + e^{\delta_q}) \prod_{i=1}^{n} \frac{\gamma_i/2}{\sinh \gamma_i/2}\right]. \end{aligned} \tag{9}$$

It is worth remarking that the product

$$\prod_{i=1}^{n} \frac{\gamma_i/2}{\sinh (\gamma_i/2)}$$

can be expressed as a power series in the Pontrjagin classes of M, which depend only on the differentiable structure, and not on the complex structure, of M. This leads to the guess that the formula may be valid for any compact complex manifold, but Hirzebruch's proof makes essential use of the fact that M is an algebraic variety.

If $q=1$, that is, if W is a line bundle which we now denote by F, the above formula reduces to

$$\chi(M, F) = \kappa_{2n}\left[e^{c_1/2+f} \prod_{i=1}^{n} \frac{\gamma_i/2}{\sinh (\gamma_i/2)}\right], \tag{10}$$

where f is the characteristic class of F. If, moreover, F is a product bundle, then $f=0$, and we have, by writing $\chi(M)$ for $\chi(M, F)$,

$$\chi(M) = \kappa_{2n}\left[e^{c_1/2} \prod_{i=1}^{n} \frac{\gamma_i/2}{\sinh (\gamma_i/2)}\right]. \tag{11}$$

The number in the right-hand side, which we shall denote by $T(M)$, was first introduced by Todd [46] and is called the Todd genus. On the other hand, we have $H^i(M, \Omega) = H^0(M, \Omega^i)$, so that the dimension g_i of $H^i(M, \Omega)$ is the dimension of the complex vector space of holomorphic i-forms, and we have

$$\chi(M) = 1 - g_1 + g_2 + \cdots + (-1)^n g_n. \tag{12}$$

$\chi(M)$ is related to the arithmetic genus $p_a(M)$ of M by the formula

$$1 + (-1)^n p_a(M) = \chi(M). \tag{12a}$$

Formula (11) thus expresses a relationship between the arithmetic genus and the Todd genus of an algebraic variety and implies in particular that the latter is an integer.

Another particular case of (10) is the case when M is an algebraic curve. In this case we get from (10),

$$\dim |D| = \dim H^0(M, \Omega(D)) - 1 = d - p + i, \tag{13}$$

where D is a divisor, d its degree, p the genus of M, and

$$i = \dim H^1(M, \Omega(D)) = \dim |K - D| + 1$$

is the index of specialty of the divisor class $|D|$. (13) is the classical Riemann-Roch theorem. Similarly, it can be seen that (9) includes as special cases other known versions of the Riemann-Roch theorem for algebraic varieties of two and three dimensions.

An essential tool in Hirzebruch's proof of (9) is the so-called index theorem. For a manifold whose dimension is not a multiple of 4 we define its index $\tau(M)$ to be zero. If M is of dimension $4k$, the cup product $u \cup u$, for $u \in H^{2k}(M, R)$, gives rise to a nonsingular symmetric quadratic form in the vector space $H^{2k}(M, R)$, and the excess of the number of its positive eigenvalues over the number of its negative eigenvalues is called the index $\tau(M)$ of M. By using the theory of "cobordisme" of Thom, [**45**], Hirzebruch proved that

$$\tau(M^{4k}) = \kappa_{4k}(L_k(p_1, \cdots, p_k)), \tag{14}$$

where $p_1, \cdots, p_k$ are the Pontrjagin classes of M^{4k}, and L_k are certain polynomials with rational coefficients which, for $k=1, 2, 3$, are given by

$$\begin{aligned} L_1 &= \frac{1}{3} p_1, \qquad L_2 = \frac{1}{45}(7p_2 - p_1^2), \\ L_3 &= \frac{1}{945}(62p_3 - 12p_1p_2 + 2p_1^3). \end{aligned} \tag{15}$$

The proof of the index theorem (14) is based on the observation that both sides of (14) depend only on the cobordisme class in the sense of Thom. Thom's theory makes use of deep results of homotopy theory in algebraic topology. As the full force of his results is not needed here, it would be desirable to give a more direct proof of (14). But even for a four-dimensional manifold such a proof is not known.

6. **Homogeneous complex manifolds.** An important class of complex manifolds consists of the homogeneous ones, that is, those which admit a transitive group of complex automorphisms. If the manifold

M is compact, the group of all its complex automorphisms is a complex Lie group; it is semi-simple if the Euler-Poincaré characteristic of M is >0. If M is a bounded domain, the group of all its complex automorphisms is a real Lie group. Since M has then the Bergmann metric, which is a homogeneous Kählerian metric, the group of isotropy of M is compact. For a general noncompact complex manifold it is well-known that the group of all its complex automorphisms is not necessarily a Lie group. It is not known whether there exists a Lie group of complex automorphisms which acts transitively on M.

Wang determined all the compact homogeneous complex manifolds with a finite fundamental group [**47**; also **18**]. In particular, those which are simply connected can be described as follows: Let K be a compact semi-simple Lie group, and T^s a toroidal subgroup of dimension s. The centralizer Z of T^s is locally a product of a toroid and a compact semi-simple group Q. A connected closed subgroup U of K is called a C-subgroup if there is a toroidal subgroup T^s with the property that $Q\subset U\subset Z$. A simply connected compact homogeneous complex manifold is homeomorphic to a real coset space K/U, where K is a semi-simple compact Lie group and U a C-subgroup of K. Conversely, every such coset space K/U, if even-dimensional, can be given a homogeneous complex structure.

Wang's work gives many examples of compact complex manifolds, besides the algebraic varieties. These include the product of two spheres of odd dimensions >1, as first given by Calabi-Eckmann [**7**], and the even-dimensional compact Lie groups. The question of determining the simply-connected compact homogeneous complex manifolds which are algebraic is simplified by a theorem of Lichnerowicz [**34**], which states that a compact simply connected homogeneous Kähler manifold has Euler-Poincaré characteristic >0. Goto, and independently Borel and Weil, proved that this condition is sufficient [**19**]. Goto also proved that the algebraic variety is rational.

Much less is known about noncompact homogeneous complex manifolds. Borel studied the coset spaces of a real semi-simple Lie group which admit an invariant Kählerian structure [**5**]. He proved that, if the coset space is noncompact, it is an analytic fiber bundle, with Kählerian manifolds as fibers, over an Hermitian symmetric space.

Perhaps the most important noncompact homogeneous complex manifolds are the homogeneous bounded domains. In this sense the question raised by E. Cartan is of importance, as to whether a homogeneous bounded domain is always symmetric. Borel [**5**] and Koszul proved independently that this is the case if the group of all complex automorphisms is semi-simple. But the general question remains unanswered.

7. **Structures defined by infinite continuous pseudo-groups.** It has been observed recently that the notion of an infinite continuous pseudo-group in the sense of Lie and Cartan gives rise to structures which generalize the complex structure. The significance of this generalization remains to be seen, and we restrict ourselves to a few general remarks [**12**; **13**].

A transformation of an infinite pseudo-group is a coordinate transformation $(x^1, \cdots, x^n) \rightarrow (x'^1, \cdots, x'^n)$ which represents a general integral of a system of partial differential equations in n independent and n dependent variables, with the property that the inverse of a transformation and the composition of two transformations are general integrals of the same differential system. They form a pseudo-group, and not a group, because the domain and the range of these transformations are allowed to vary. If the system of partial differential equations is supposed to be of the first order, the transformations can be defined to be those which reproduce n linearly independent Pfaffian forms up to a linear transformation of a given linear group. The latter is called the group of stability. The local theory of such pseudo-groups was developed by Lie and E. Cartan. It is useful to remark that these pseudo-groups are numerous.

Corresponding to every such pseudo-group there exists a type of structure on a class of differentiable manifolds, with the property that the transition of local coordinate systems is given by a transformation of the pseudo-group. The complex structure and complex manifolds constitute a notable example of this general notion. There are, however, other structures which exist in a natural way. We mention the leaved structure of Ehresmann-Reeb and the structure on an odd-dimensional manifold with a Pfaffian equation of the maximum class given everywhere. The latter structure exists on the manifold of tangent covariant directions of a differentiable manifold. (A covariant direction is the class of all nonzero covariant vectors which differ from each other by a factor.) A structure-preserving automorphism on such an odd-dimensional manifold is essentially what is known as a contact transformation.

Many notions which have been developed in the theory of the complex structure can be generalized. Thus manifolds for which the structural group of the tangent bundle can be reduced to the group of stability of an infinite pseudo-group generalize the almost complex manifolds. There is also a natural generalization of the Kähler property [13]. We can also define a class of manifolds analogous to the algebraic varieties. In fact, according to a theorem of Chow, the latter are complex manifolds which can be complex analytically imbedded in a complex projective space. The complex projective space is unique among the complex manifolds in the sense that it is the only simply connected, compact complex manifold with a group of complex automorphisms which is transitive on the tangent directions. If certain "universal spaces" of our generalized structure can be defined, universal in the sense that they admit a "large" group of structure-preserving automorphisms, then the analogues of algebraic varieties will be the manifolds imbeddable in universal spaces with structures preserved.

Finally, we remark that the sheaves will again be a useful tool in the study of such manifolds. Since the structural group of the tangent bundle can be restricted to the group of stability G and the representation of G in the space of alternating tensors may be reducible, we can speak of exterior differential forms of a type corresponding to any invariant subspace of such a representation. The sheaf of germs of differential forms of a given type is thus well defined. Moreover, from the exterior differentiation operator and projections of a differential form into one of a given type, various differential operators can be defined. This leads to the definition of cohomology groups whose study should be of importance in the theory of manifolds with structures in the sense we have explained.

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SHIING-SHEN CHERN

PART III. ALGEBRAIC SHEAF THEORY

The cohomological methods, *in conjunction with the powerful tool of harmonic integrals*, were remarkably effective in the solution of global complex-analytic problems in general, and of problems of classical algebraic geometry in particular (Chern, Hirzebruch, Kodaira-Spencer, Serre, and others). It is natural to ask whether the cohomological methods can be equally effective in abstract algebraic geometry where the method of harmonic integrals is no longer available.