33. ———, *On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 1273–1278.

34. A. Lichnerowicz, *Espaces homogènes Kählériens*, Colloques Strassbourg (1953) pp. 172–184.

35. J. C. Moore, H. K. Nickerson and D. C. Spencer, *Mimeographed notes on sheaves and differentiable manifolds*, Princeton, 1954.

36. S. Nakano, *On a certain type of analytic fiber bundles*, Proc. Japan Acad. vol. 30 (1954) pp. 542–547.

37. ———, *On complex analytic vector bundles*, J. Math. Soc. Japan vol. 7 (1955) pp. 1–12.

38. G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Scientifiques et Industrielles, no. 1183, 1952.

39. J-P. Serre, *Quelques problèmes globaux relatifs aux variétés de Stein*, Colloques de Bruxelles (1953) pp. 57–68.

40. ———, *Un théorème de dualité*, Comment. Math. Helv. vol. 29 (1955) pp. 9–26.

41. ———, *Faisceaux algébriques cohérents*, Ann. of Math. vol. 61 (1955) pp. 197–278.

42. N. Steenrod, *Fibre bundles*, Princeton University Press, 1951.

43. K. Stein, *Analytische Funktionen mehrer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem*, Math. Ann. vol. 123 (1951) pp. 201–222.

44. W. Stoll, *Über meromorphe Modifikationen*, Math. Zeit. vol. 61 (1954) pp. 206–234.

45. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment Math. Helv. vol. 28 (1954) pp. 17–86.

46. J. A. Todd, *The geometrical invariants of algebraic loci*, Proc. London Math. Soc. (2) vol. 43 (1937) pp. 127–138.

47. H. C. Wang, *Closed manifolds with homogeneous complex structure*, Amer. J. Math. vol. 76 (1954) pp. 1–32.

48. A. Weil, *Fiber spaces in algebraic geometry*, Notes by A. Wallace, University of Chicago, 1952.

49. Wen-Tsun Wu, *Sur les classes caractéristiques des structures fibrées sphériques*, Actualités Scientifiques et Industrielles, no. 1183, 1952.

50. O. Zariski, *Algebraic surfaces*, Berlin, Springer, 1935.

51. F. Hirzebruch, *Neue topologische methoden in der olgebraischen Geometrie*, Springer, 1956.

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PART III. ALGEBRAIC SHEAF THEORY

The cohomological methods, *in conjunction with the powerful tool of harmonic integrals*, were remarkably effective in the solution of global complex-analytic problems in general, and of problems of classical algebraic geometry in particular (Chern, Hirzebruch, Kodaira-Spencer, Serre, and others). It is natural to ask whether the cohomological methods can be equally effective in abstract algebraic geometry where the method of harmonic integrals is no longer available.

The purely algebraic theory of sheaves developed by Serre [3][1] represents the first systematic application of cohomological algebra to abstract algebraic geometry. The theory developed by Serre includes the "generalized lemma of Enriques-Severi" proved and so named by Zariski [4], and contains also the first algebraic proof of Severi's conjecture $P_a = p_a$. These are highly encouraging indications of the power and potentialities of the cohomological method in abstract algebraic geometry.[2]

1. **The general concept of a sheaf.** A *sheaf* $\mathcal{J}$ is a composite concept consisting of two topological spaces F and X and a continuous mapping (*projection*) π of F *onto* X satisfying the following conditions: (1) π is a local homeomorphism; (2) for each $x \in X$ the set $\mathcal{J}_x = \pi^{-1}\{x\}$ is an abelian (additive) group; (3) the group structure of $\mathcal{J}_x$ varies continuously with x. The space X is called the *base space* of the sheaf, and $\mathcal{J}_x$ is called the *stalk* over x. If U is an open subset of X, a *section* of $\mathcal{J}$ over U is a continuous mapping f of U into F such that πf is the identity on U. The sections of $\mathcal{J}$ over U form in an obvious way an abelian (additive) group, denoted by $\Gamma(\mathcal{J}, U)$. As U ranges over an open basis of X and f ranges over $\Gamma(\mathcal{J}, U)$, the sets $f(U)$ yield an open basis of F, as follows easily from the definitions. An important property of sections is the following: if $f, g \in \Gamma(\mathcal{J}, U)$ and $f(x) = g(x)$ for some $x \in U$, then $f = g$ on some open neighborhood of x. In practice, one often introduces a sheaf $\mathcal{J}$ as the union F of its stalks $\mathcal{J}_x$ ($x \in X$), and the topology of F is then defined by assigning the local sections of $\mathcal{J}$.

The general notions of a subsheaf of a given sheaf $\mathcal{J}$, factor sheaf, sheaf homomorphism, direct sum and tensor product of sheaves (with

[1] Numbers in brackets refer to the bibliography at the end of this part of the report.

[2] The participants of the seminar on algebraic sheaf theory had in their possession a short manuscript of Serre (Serre [2]) in which he gave the definitions and stated (almost always without proof) the basic results of the theory. It seemed fruitful to take this manuscript as the subject of the seminar and to try to reconstruct the proofs, this being the best way of acquainting ourselves with the new algebraic methods, ideas and results. The present report is based:

(1) on the work of that seminar;

(2) on manuscripts given to the writer of this report by J. Igusa and S. Lang (who, together with G. Washnitzer, were the most active members of the seminar);

(3) on Serre's Annals paper [3] which became subsequently available and which formed the subject of a seminar conducted at Harvard in 1954–1955 by Igusa and the writer.

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the same base space) are defined in a natural fashion (Cartan [3], Serre [3]). It is important to bear in mind that, by definition, a subsheaf $\mathcal{J}'$ of $\mathcal{J}$ is always open in F and that the canonical sheaf homomorphism $\mathcal{J}\to\mathcal{J}/\mathcal{J}'$ is always a continuous *open* mapping. A sheaf epimorphism[3] j: $\mathcal{J}\to\mathcal{J}''$ induces a homomorphism j_x of $\mathcal{J}_x$ onto $\mathcal{J}_x''$ for each x in X and also a homomorphism j_U of $\Gamma(\mathcal{J}, U)$ into $\Gamma(\mathcal{J}'', U)$ for each open subset U of X. Furthermore, each section of $\mathcal{J}''$ is *locally* the j-image of a section of $\mathcal{J}$. More precisely: if $f''\in\Gamma(\mathcal{J}'', U)$ and $x\in U$, there exists a neighborhood W of x, contained in U, and there exists a section f in $\Gamma(\mathcal{J}, W)$ such that $jf=f''$ on W.

In applications to algebraic geometry the stalks $\mathcal{J}_x$ of a sheaf possess, as a rule, much more structure than that of mere abelian groups. The stalks may be rings, and in that case the ring structure of $\mathcal{J}_x$ is assumed to vary continuously with x. Or, each $\mathcal{J}_x$ may be a module over some fixed field. Or finally—and this is the basic setup in applications to algebraic geometry—we may deal with sheaves $\mathcal{J}$ which admit a fixed basis sheaf $\mathcal{O}$ as *sheaf of operators*: each stalk $\mathcal{O}_x$ is then a ring, and $\mathcal{J}_x$ is a module over $\mathcal{O}_x$, the ring structure of $\mathcal{O}_x$ and the module structure of $\mathcal{J}_x$ both varying continuously with x.

2. **Examples of sheaves.** (a) Let X be a complex analytic manifold and let $\tilde{\mathcal{O}}_x$ be the ring of germs of holomorphic functions at x ($x\in X$). A section over an open set U is then described by a function f which is holomorphic on U (i.e., by the mapping $x\to f_x$, where $x\in U$ and f_x is the germ of f at x). This is the fundamental sheaf of the classical theory.

(b) Let us take for X an irreducible algebraic variety V in an affine or projective space, over some algebraically closed ground field k. We must first decide on the choice of a topology in V. In the classical case the choice is obvious: we take the ordinary Hausdorff topology of V. However, V also carries another topology, introduced by Zariski. In this "Zariski topology," the closed sets are the algebraic subvarieties of V. The Zariski topology is much weaker than the usual Hausdorff topology but it has a perfect meaning in the abstract case. Serre took the Zariski topology for his algebraic sheaf theory. At the same time he took as fundamental sheaf the sheaf $\mathcal{O}=\mathcal{O}_V$ whose stalks $\mathcal{O}_x$ are the local rings of V at the various points x of V. Local sections of $\mathcal{O}$ are then described by locally regular functions f which are meromorphic on V; any such function is an element of the function field K of V. The points x of V where f is not regular (i.e.,

[3] We shall use the following terminology: a homomorphism j: $A\to A'$ is an *epimorphism* if j maps A *onto* A'; it is a *monomorphism* if its kernel is zero; and it is an *isomorphism* if it is both an epimorphism and a monomorphism.

the points x such that $f \notin \mathcal{O}_x$) form an algebraic subvariety W of V, and thus f defines a section of $\mathcal{O}$ over $U = V - W$ (U is open in the Zariski topology).

A sheaf on V (i.e., a sheaf having V as base space) is *algebraic* if it admits $\mathcal{O}$ as sheaf of operators. These are the sheaves studied by Serre. A homomorphism j between algebraic sheaves $\mathcal{J}$ and $\mathcal{J}''$ is called *algebraic* if it is $\mathcal{O}$-linear, i.e., if for every x in V the local homomorphism j_x is $\mathcal{O}$-linear.

The sheaf $\mathcal{O}$ itself is algebraic. What are the global sections of $\mathcal{O}$? The answer depends on whether V is an affine or projective variety. If V is an affine variety in r-space, then $\Gamma(\mathcal{O}, V)$ is the (nonhomogeneous) coördinate ring $R = k[x_1, x_2, \cdots, x_r]$ of V/k. If V is a projective variety then $\Gamma(\mathcal{O}, V) = k$.

Another example of an algebraic sheaf is the p-fold direct product of $\mathcal{O}$, denoted by $\mathcal{O}^p$. The elements of $\mathcal{O}^p_x$ are then the ordered p-tuples $(\alpha_1, \alpha_2, \cdots, \alpha_p)$, $\alpha_i \in \mathcal{O}_x$. These sheaves play an important role in the theory of coherent sheaves (see next section).

It should be noted that if k is the field of complex numbers we have associated two types of sheaves with a variety V, namely $\tilde{\mathcal{O}}$ and $\mathcal{O}$. They differ in two important respects: in the topology of their base space V and in the fact that $\tilde{\mathcal{O}}_x$ contains $\mathcal{O}_x$ but is a much bigger ring then $\mathcal{O}_x$. Therefore in the classical case we have *a priori* two sheaf theories, or two cohomology theories of V: one is analytic, the other is algebraic. Serre has established an "isomorphism" between these two theories (see end of §7).

(c) Let V be again an irreducible algebraic variety and let W be a fixed affine or projective subvariety of V. Zariski has defined in his memoir [**3**] the notion of a function on V which is holomorphic along an open subset U of W. *We take W as base space* and we take as stalk $\mathcal{O}_x^*(x \in W)$ the set of germs, at x, of functions f on V which are holomorphic along some open subset of W containing x. If we assume that W is irreducible and that V is analytically irreducible at each point of W (this latter condition is satisfied, for instance, if V is normal), then it is known that the germ of a function f at x determines the function uniquely (Zariski [**3**]). This enables us to define a sheaf $\mathcal{O}^*_{V,W}$ by taking the union of the stalks $\mathcal{O}_x^*$. It may be of interest to examine the results and the unsolved questions of Zariski's memoir [**3**] from the standpoint of sheaf theory, by developing the properties of sheafs on W which admit $\mathcal{O}^*_{V,W}$ as sheaf of operators.

3. Coherent sheaves. An algebraic sheaf $\mathcal{J}$ on a variety V is *coherent* if locally, i.e., in the neighborhood of each point x of V, $\mathcal{J}$ can be written in the form $\mathcal{O}^p/\phi(\mathcal{O}^q)$, where p and q are non-negative integers and ϕ is an algebraic homomorphism of $\mathcal{O}^q$ into $\mathcal{O}^p$ (p, q and ϕ depend on x). In other words, in the neighborhood of any point x of V we have an exact sequence:

$$\mathcal{O}^q \xrightarrow{\phi} \mathcal{O}^p \xrightarrow{j} \mathcal{J} \to 0, \tag{1}$$

all sheaf homomorphisms being assumed algebraic. If U denotes a neighborhood in which (1) holds and if $\mathcal{J}_U$ denotes the sheaf $\bigcup \mathcal{J}_x$, $x \in U$, with U as base space ($\mathcal{J}_U$ = *restriction of* $\mathcal{J}$ *to* U), then the precise way of writing (1) is the following:

$$\mathcal{O}_U^q \xrightarrow{\phi} \mathcal{O}_U^p \xrightarrow{j} \mathcal{J}_U \to 0. \tag{1'}$$

Recalling from §1 that the sections of a homomorphic image of a given sheaf (in this case, of the image of $\mathcal{O}_U^p$) are locally images of sections of the given sheaf, the existence of the homomorphism j of $\mathcal{O}_U^p$ onto $\mathcal{J}_U$ is equivalent with the following property of $\mathcal{J}$: *for each point x of V there exists an open set W containing x and there exists a finite set of sections $f_1, f_2, \cdots, f_p$ of $\mathcal{J}$ over W, such that for each point y of W the elements $f_1(y), f_2(y), \cdots, f_p(y)$ form a module basis of $\mathcal{J}_y$ over $\mathcal{O}_y$* (or briefly: the sections $f_1, f_2, \cdots, f_p$ generate the stalk $\mathcal{J}_y$ for all y in W). An algebraic sheaf $\mathcal{J}$ is therefore coherent if and only if both $\mathcal{J}$ *and* the kernel of j have this property. This property shows that the theory of coherent sheaves is an additive theory, and that the modules entering in this theory are expected to be of finite type. It turns out that most sheaves which one encounters in applications to algebraic geometry (for instance, the sheaves which one must consider in connection with the Riemann-Roch theorem) are indeed coherent. It should be noted, however, that there are interesting sheaves which are not coherent, for instance the sheaf of local units: the stalk at each point x of V consists of the units of $\mathcal{O}_x$. In the classical theory this sheaf plays an important role which up to now has no counterpart in the abstract theory.

If $f_1, f_2, \cdots, f_m$ are sections of a sheaf $\mathcal{J}$ over a set U, one denotes by $\text{Rel}_x (f_1, f_2, \cdots, f_m)(x \in U)$ the set of m-tuples $(a_1, a_2, \cdots, a_m)$, $a_i \in \mathcal{O}_x$, such that $a_1f_1(x) + a_2f_2(x) + \cdots + a_mf_m(x) = 0$, and by $\text{Rel}_U (f_1, f_2, \cdots, f_m)$ that subsheaf of $\mathcal{O}_U^m$ whose stalks are the modules $\text{Rel}_x (f_1, f_2, \cdots, f_m)$. Thus, returning to (1′) and with the same notations as above, $\text{Rel}_U (f_1, f_2, \cdots, f_p)$is the kernel of j.

The three lemmas stated below are auxiliary results which are frequently used in the theory of coherent sheaves:

LEMMA 1. *If $\mathcal{J}$ is a coherent sheaf on V and $f_1, f_2, \cdots, f_m$ are sections of $\mathcal{J}$ over an open set U, then the sheaf* $\text{Rel}_U (f_1, f_2, \cdots, f_m)$ *is coherent.*

LEMMA 2. *If j is an algebraic homomorphism of a coherent sheaf $\mathcal{J}$ into a coherent sheaf $\mathcal{G}$, then the kernel $\mathcal{J}'$, the image $\mathcal{J}''=j(\mathcal{J})$ and the co-kernel $\mathcal{G}/\mathcal{J}''$ of j are coherent sheaves.*

Lemma 1 is analogous to the theorem of Oka-Cartan (Cartan [**3**, pp. 3–18, 1951–1952]). Lemma 2 is a simple consequence of Lemma 1.

By a *principal open* subset U of an affine variety V we mean a subset of V whose complement (boundary) $V-U$ is the set of zeros of a function Q belonging to the coördinate ring R of V. It is easily seen that a principal open set is again biregularly equivalent to an affine variety and that the set of coverings of a variety (projective or affine) with principal open sets is cofinal with the set of all open coverings of the variety.

LEMMA 3. *If a global section f of a coherent sheaf $\mathcal{J}$ on an affine variety V is zero at each point of a principal open subset U of V and if $Q=0$ is an equation of the boundary of U, then for some integer m we have $Q^m f=0$ everywhere on V.*

The self-evident local character of this lemma allows a reduction of the proof to the special case in which $\mathcal{J}$ is isomorphic with $\mathcal{O}_V^n/\phi(\mathcal{O}_V^m)$ (ϕ, an algebraic homomorphism) and f is the image of a global section of $\mathcal{O}_V^n$. In this special case, the lemma is an easy consequence of the Hilbert Nullstellensatz.

The above definitions and lemmas hold both for affine and projective varieties, and indeed they remain valid for "abstract varieties" in the sense of Weil. While the ultimate goal of the cohomological theory of sheaves lies in the direction of projective varieties (or even "abstract varieties" in the sense of Weil), a preliminary study of sheaves on affine varieties is essential, since the set of open coverings of a projective variety V by affine varieties is cofinal with the set of all open coverings of V. The first part of the seminar was therefore entirely devoted to coherent sheaves on affine varieties.

4. Cohomology groups and exact cohomology sequences. Returning to the general theory of an arbitrary topological space X and an arbitrary sheaf $\mathcal{J}$ over X, one can define cohomology groups $H^q(X,\mathcal{J})$ of X, with coefficients in $\mathcal{J}$, by the Čech methods, as follows:

If $\mathfrak{U}=\{U_i\}$ is a finite open covering of X by open sets U_i, a q-cochain f, with coefficients in $\mathcal{J}$, is a function which associates with each q-simplex $\sigma=\{U_{i_0}, U_{i_1}, \cdots, U_{i_q}\}$ of the covering $\mathfrak{U}$ a section $f_{i_0,i_1,\cdots,i_q}$ of $\mathcal{J}$ over the support (sup σ) $U_{i_0}\cap U_{i_1}\cap\cdots\cap U_{i_q}$ of σ (it is assumed that f is an alternating function of the indices i_0, $i_1, \cdots, i_q$). The set of all q-cochains (relative to the given covering

$\mathfrak{U}$) is denoted by $C^q(\mathfrak{U}, \mathcal{J})$. One defines the coboundary δf by the usual formula

$$(\delta f)_{i_0 i_1 \cdots i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j f_{i_0 i_1 \cdots \hat{i}_j \cdots i_{q+1}},$$

where the sections of $\mathcal{J}$ on the right must be intended actually as restriction maps of $U_{i_0} \cap U_{i_1} \cap \cdots U_{i_{q+1}}$. This yields cochains, cocycles and cohomology groups $H^q(\mathfrak{U}, \mathcal{J})$ depending on the covering $\mathfrak{U}$, *except that it is immediately seen that* $H^0(\mathfrak{U}, \mathcal{J})$ *is isomorphic with* $\Gamma(\mathcal{J}, X)$ and is thus independent of the covering $\mathfrak{U}$. One then passes to the inductive limit of the groups $H^q(\mathfrak{U}, \mathcal{J})$ on the directed set of all finite open coverings of X and one thus obtains the cohomology groups $H^q(X, \mathcal{J})$.

Suppose that we have an exact sequence of sheaves (over the same base space X):

$$0 \to \mathcal{J}' \to \mathcal{J} \to \mathcal{J}'' \to 0. \tag{1}$$

Then for any open covering $\mathfrak{U}$ of X we find immediately a corresponding exact sequence

$$0 \to C^q(\mathfrak{U}, \mathcal{J}') \xrightarrow{i} C^q(\mathfrak{U}, \mathcal{J}) \xrightarrow{j} C^q(\mathfrak{U}, \mathcal{J}''). \tag{2}$$

In general, j is not necessarily onto, so that we cannot complete the sequence (2) with a 0 at the end.

In the classical case, where we are dealing with spaces X which are paracompact and Hausdorff, one can nevertheless obtain a corresponding exact sequence of the cohomology groups $H^q(X, \mathcal{J})$:

$$\begin{aligned} &0 \to H^0(X, \mathcal{J}') \to \cdots \to H^q(X, \mathcal{J}') \to H^q(X, \mathcal{J}) \to H^q(X, \mathcal{J}'') \\ &\quad \to H^{q+1}(X, \mathcal{J}') \to \cdots . \end{aligned} \tag{3}$$

A proof can be found in mimeographed Princeton notes, Chapter 1, *Sheaves*, by J. C. Moore. The following outline of a simplified proof has been communicated by D. C. Spencer and is based on the treatment of Godement-Serre (mimeographed notes of Godement's lectures at the University of Illinois, 1954–1955, and Serre [**3**]).

We shall use the notion of a "pre-sheaf" (Leray sheaf). A pre-sheaf assigns to each open set U of the topological space X a K-module $A(U)$, K—a principal ideal ring. If U and V are open sets and $V \subset U$, there is a K-homomorphism ρ_{VU}: $A(U) \to A(V)$ such that for $W \subset V \subset U$ we have the transitivity condition $\rho_{WV}\rho_{VU} = \rho_{WU}$. Note that any pre-sheaf defines, by passing to limts, a sheaf over X.

A homomorphism $\phi: A \to B$ of the pre-sheaf A into the pre-sheaf B is, for each open U, a K-homomorphism $A(U) \to B(U)$ which commutes with the restriction homomorphism ρ_{VU}. A sequence $0 \to A' \to A \to A'' \to 0$ of pre-sheaf homomorphism is said to be exact if and only if $0 \to A'(U) \to A(U) \to A''(U) \to 0$ is exact for every open U. It follows that if we have an exact sequence $0 \to A' \to A \to A'' \to 0$ of pre-sheaf homomorphisms and if $\mathfrak{U}$ is an open covering of X, then we also have the following exact sequence

$$0 \to C^q(\mathfrak{U}, A') \xrightarrow{i} C^q(\mathfrak{U}, A) \xrightarrow{j} C^q(\mathfrak{U}, A'') \to 0, \tag{4}$$

where the group $C^q(\mathfrak{U}, A)$ of q-cochains with coefficients in a pre-sheaf is defined in the same way as in the case of sheaves. Then by a standard construction of algebraic topology there follows from the exactness of (4) the existence of an exact cohomology sequence:

$$\begin{aligned} &0 \to H^0(\mathfrak{U}, A') \xrightarrow{i^*} H^0(\mathfrak{U}, A) \xrightarrow{j^*} \cdots \to H^q(\mathfrak{U}, A') \\ &\xrightarrow{i^*} H^q(\mathfrak{U}, A) \xrightarrow{j^*} H^q(\mathfrak{U}, A'') \xrightarrow{\delta^*} H^{q+1}(\mathfrak{U}, A') \to \cdots . \end{aligned} \tag{5}$$

Passing to the inductive limit we obtain a corresponding exact sequence of cohomology groups $H^q(X, A)$:

$$\begin{aligned} &0 \to H^0(X, A') \to \cdots \to H^q(X, A') \to H^q(X, A) \to H^q(X, A'') \\ &\to H^{q+1}(X, A') \to \cdots . \end{aligned} \tag{6}$$

One proves quite simply the following result:

Let X be paracompact and Hausdorff and let N be a pre-sheaf such that: (a) $N(U)=0$ if $U=\emptyset$; (b) the sheaf defined by N is zero. Then $H^q(X, N)=0$ for all $q \geqq 0$. As a corollary we obtain the result that, if $\phi: A \to A'$ is a homomorphism of pre-sheaves which induces an isomorphism of the corresponding sheaves $\mathcal{J}$, $\mathcal{J}'$ then $H^q(X, \mathcal{J}) \cong H^q(X, \mathcal{J}')$ for all $q \geqq 0$.

Now consider an exact sequence $0 \to \mathcal{J}' \to^i \mathcal{J} \to^j \mathcal{J}'' \to 0$ of homomorphisms of *sheaves*. The modules $\Gamma(\mathcal{J}', U)$, $\Gamma(\mathcal{J}, U)$ and $\Gamma(\mathcal{J}'', U)$ define pre-sheaves A', A and A'', and we have an exact sequence

$$0 \to \Gamma(\mathcal{J}', U) \to \Gamma(\mathcal{J}, U) \to \Gamma_0(\mathcal{J}'', U) \to 0,$$

where $\Gamma_0(\mathcal{J}'', U)$ denotes the image of $\Gamma(\mathcal{J}, U)$ in $\Gamma(\mathcal{J}'', U)$. Clearly, also the modules $\Gamma_0(\mathcal{J}'', U)$ define a pre-sheaf, which we shall denote by A_0'', so that we have the exact sequence $0 \to A' \to A \to A_0'' \to 0$. We have furthermore the exact sequence $0 \to A_0'' \to A'' \to Q \to 0$, i.e.,

$$0 \to \Gamma_0(\mathcal{J}'', U) \to \Gamma(\mathcal{J}'', U) \to Q(U) \to 0, \tag{7}$$

where the *pre-sheaf Q defines in the limit the zero sheaf* (this is so because locally each section of $\mathcal{J}''$ is the j-image of a section of $\mathcal{J}$). From the exactness of the pre-sheaf homomorphisms (7), we find, using (6) and observing that $H^q(X, A'') \cong H^q(X, \mathcal{J}'')$:

$$H^q(X, A_0'') \cong H^q(X, A'') \cong H^q(X, \mathcal{J}'').$$

Therefore, applying (6) to the exact sequence $0 \to A' \to A \to A_0''$, we find the desired cohomology sequence (3), since $H^q(X, A') \cong H^q(X, \mathcal{J}')$ and $H^q(X, A) \cong H^q(X, \mathcal{J})$.

Since the existence of the exact sequence (3) is the key to all the applications of sheaf theory, it is essential to establish (3) for *coherent sheaves* also in the abstract case. In the abstract case, however, it turns out that *even the sequence* (2) *can be completed with a zero at the end*. The problem then is to prove that j maps $C^q(\mathfrak{U}, \mathcal{J})$ *onto* $C^q(\mathfrak{U}, \mathcal{J}'')$, if $\mathcal{J}$ and $\mathcal{J}''$ are coherent sheaves, and it is sufficient to prove this only for coverings $\mathfrak{U}$ by principal open sets. If $\mathfrak{U}$ is such a covering, then the support $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q}$ of every simplex of the nerve of $\mathfrak{U}$ is again a principal open set, hence is biregularly equivalent to an affine variety. Thus, our problem is therefore to prove the following theorem:

THEOREM 1. *If j is an algebraic homomorphism of a coherent sheaf $\mathcal{J}$ onto a coherent sheaf $\mathcal{J}''$ on an affine variety V, then j maps $\Gamma(\mathcal{J}, V)$ onto $\Gamma(\mathcal{J}'', V)$.*

In the proof of this theorem there are three distinct stages which can be summarized as follows:

1. First one proves directly Theorem 1 in the special case $\mathcal{J} = \mathcal{O}_V^m$. The proof in this case is a simple application of Lemma 3 (§3), in conjunction with the fact that *locally* the sections of $\mathcal{J}''$ are j-images of sections of $\mathcal{J}$ (§1).

2. Using the above special case of Theorem 1 one obtains easily the following important auxiliary result:

LEMMA 4. *If $\mathcal{J}$ is the homomorphic image of $\mathcal{O}_V^m$ (V—an affine variety) and f is a section of $\mathcal{J}$ over a principal open subset U of V having boundary $Q = 0$ (Q—an element of the coordinate ring of V), then for sufficiently high integers q the section $Q^q f$ can be extended to a section of $\mathcal{J}$ over V.*

Actually, Lemma 4 already has to be used at Step 1, but only in the special case $\mathcal{J} = \mathcal{O}_V^m$, and in that case the lemma is trivial.

3. Finally, using Lemma 4 and the fact that *locally* every coherent sheaf is of the form $\mathcal{O}^m/\phi(\mathcal{O}^q)$, one proves the following basic result:

THEOREM 2. *If $\mathcal{F}$ is a coherent sheaf on an affine variety V, then there exists an exact sequence*:

$$\mathcal{O}_V^q \to \mathcal{O}_V^m \to \mathcal{F} \to 0.$$

Thus, for affine varieties, the local conditions which define coherent sheaves (see (1), §3) can be replaced by *one* similar *global* condition. At any rate, every coherent sheaf on an affine variety V is thus a homomorphic image of $\mathcal{O}_V^m$, for some m.

Theorem 1 now follows easily. For, we have a homomorphism α of $\mathcal{O}_V^m$ onto $\mathcal{F}$, for some m, and thus $j\alpha$ is a homomorphism of $\mathcal{O}_V^m$ onto $\mathcal{F}''$. If $f'' \in \Gamma(\mathcal{F}'', V)$, we have, by the special case of Theorem 1, proved in step 1, that $f'' = (j\alpha)(g)$, for some g in $\Gamma(\mathcal{O}_V^m, V)$. If we set $f = \alpha(g)$, then $f'' = j(f)$ and $f \in \Gamma(\mathcal{F}, V)$.

The so-called "partition of unity" which played an important role in the classical theory has the following algebraic analogue which is also used in the abstract case (and in the proofs of Theorems 1 and 2): *If $\mathfrak{U}$ is a finite covering of an affine variety V by principal open subsets U_i and if $Q_i = 0$ is an equation of the boundary of $U_i (Q_i \in R)$, then there exist elements G_j in R such that* $\sum G_i Q_i = 1$. This is another formulation of the Hilbert Nullstellensatz.

5. **Cohomological properties of coherent sheaves on affine varieties.** A sheaf $\mathcal{F}$ on a variety V is said to be *semifine* (Serre [1]) if it satisfies the following two conditions:

SF.1. $H^0(V, \mathcal{F})$ (i.e., $\Gamma(\mathcal{F}, V)$) generates the stalk $\mathcal{F}_x$ over $\mathcal{O}_x$ at every point x of V.

SF.2. $H^q(V, \mathcal{F}) = 0$ if $q > 0$.

Theorem 2 (§4) shows that every coherent sheaf $\mathcal{F}$ over an *affine* variety V satisfies condition SF.1. In fact, if $\mathcal{F}$ is the image of $\mathcal{O}_V^m$, under a homomorphism j, and $E_1, E_2, \cdots, E_m$ are the basic unit sections of $\Gamma(\mathcal{O}_V^m, V)$, the m global section $j(E_i)$ of $\mathcal{F}$ generates $\mathcal{F}_x$ over $\mathcal{O}_x$ at every point x. Note that $E_1, E_2, \cdots, E_m$ generate $\Gamma(\mathcal{O}_V^m, V)$ over the coördinate ring R of $V(R = \Gamma(\mathcal{O}_V, V))$. Since, by Theorem 1, j induces a R-homomorphism of $\Gamma(\mathcal{O}_V^m, V)$ *onto* $\Gamma(\mathcal{F}, V)$, it follows that $j(E_1), j(E_2), \cdots, j(E_m)$ generate $\Gamma(\mathcal{F}, V)$ over R. Thus, Theorems 1 and 2 show that *$H^0(V, \mathcal{F})$ is finitely generated over R* (V—affine, $\mathcal{F}$—coherent).

If we apply the above argument to any subset U of V such that U itself is biregularly equivalent to an affine variety, we find that the restrictions of the $j(E_i)$ to U generate $\Gamma(\mathcal{F}, U)$ over $\Gamma(\mathcal{O}_V, U)$. We have thus found that every coherent sheaf $\mathcal{F}$ on an affine variety V enjoys the following property: *there exists a finite number of global sections $f_1, f_2, \cdots, f_m$ of $\mathcal{F}$ such that for any open subset U of V which* *is biregularly equivalent to an affine variety* (*in particular, for any principal open subset* U *of* V) *the restrictions of the* f_i *to* U *generate* $\Gamma(\mathcal{J}, U)$ *over the coördinate ring* $\Gamma(\mathcal{O}_V, U)$ *of* U. From this property, and using the algebraic analogue of the "partition of unity" for affine varieties (§4), it follows by a well-known argument (Weil [**2**]) that *if* $\mathfrak{U}$ *is a finite covering of an affine variety* V *by principal subsets* U_i, *then* $H^q(\mathfrak{U}, \mathcal{J})=0$ *for* $q>0$. Since the set of finite coverings of V by principal subsets is co-final with the set of all open coverings of V, it follows that $H^q(V, \mathcal{J})=0$ if $q>0$, and this is condition SF.2. We have therefore

THEOREM 3. *Every coherent sheaf* $\mathcal{J}$ *on an affine variety is semi-fine, and for such a sheaf we have that* $H^0(V, \mathcal{J})$ *is finitely generated over the coördinate ring* R *of* V (cf. Cartan [**3**, Exp. 18–19, 1951–1952]).

If we set $M=H^0(V, \mathcal{J})$, then M is a finitely generated R-module, and it generates $\mathcal{J}_x$ at every point x of V. Conversely, starting with any finite R-module M we can define a sheaf $\mathcal{J}$ over V by setting $\mathcal{J}_x=M_R\times\mathcal{O}_x$ (= tensor product, over the common ring of operators R) and by defining local sections in a natural manner, by means of the local sections of $\mathcal{O}_V$. One finds then that there is a (1, 1) correspondence between finite R-modules and coherent sheaves over V.

All these results (except the property of $H^0(V, \mathcal{J})$ of being finitely generated) are analogous to well-known statements concerning Stein varieties (see Part I of this report) and coherent sheaves over them. However, the proofs for affine varieties are much easier, and, as we have seen, are essentially of elementary nature.

Using a well-known theorem on "double complexes" (Leray [**1**]; H. Cartan [**3**, 1953–1954]), one derives at once from Theorem 3 the following result (Serre [**3**]; for the classical case see Leray [**1**], Weil [**2**]):

THEOREM 4. *If* $\mathfrak{U}$ *is a finite covering of a variety* V *by affine varieties* U_i *and* $\mathcal{J}$ *is a coherent sheaf on* V, *then the canonical homomorphism of* $H^q(\mathfrak{U}, \mathcal{J})$ *into* $H^q(V, \mathcal{J})$ *is an isomorphism.*

Theorem 4 shows that to catch the cohomology group of an arbitrary variety V, relative to a given coherent sheaf, it is not necessary to pass to the inductive limit: those groups which are obtained from the nerve of a covering of V by affine varieties are already isomorphic to the inductive limit. The cohomology groups of V (in the case of coherent sheaves) can therefore be found by purely algebraic processes.

6. Coherent sheaves on projective varieties. A first consequence of Theorem 4 is the following: *if* $\mathcal{J}$ *is a coherent sheaf on a projective* *variety* V, *of dimension* m, *then* $H^q(V, \mathcal{F})=0$ *for* $q>m$. In view of Theorem 4 it is sufficient to show that $H^q(\mathfrak{U}, \mathcal{F})=0$, $q>m$ and for some finite covering $\mathfrak{U}$ of V by affine varieties. We fix $m+1$ hyperplanes H_i which have no point in common on V. Then $\mathfrak{U}=\{V-H_i\}$ is a covering of V by principal open sets, and since the nerve of $\mathfrak{U}$ has only $m+1$ vertices it follows that $H^q(\mathfrak{U}, \mathcal{F})=0$ for $q>m$.—Before Serre, this result was derived from *Dolbeault's isomorphism* (Dolbeault [1]), but for locally free sheaves only.

The treatment of sheaves on projective varieties is greatly facilitated by the operation of extension of sheaves. This operation allows us to replace any sheaf $\mathcal{F}$ over a variety V by a sheaf $\mathcal{G}$ carried by the ambient projective space X of V: we simply set $\mathcal{G}_x=\mathcal{F}_x$ for $x\in V$ and $\mathcal{G}_x=0$ if $x\notin V$. Conversely, if a sheaf $\mathcal{G}$ over X is such that $\mathcal{G}_x=0$ outside of a variety V, then $\mathcal{G}$ gives rise to a unique sheaf $\mathcal{F}$ carried by V. We shall often identify sheaves $\mathcal{F}$ and $\mathcal{G}$ in such a relation, since it is easily seen that $H^q(V, \mathcal{F})=H^q(X, \mathcal{G})$ for all q. *If* $\mathcal{F}$ *is algebraic then* $\mathcal{G}$ *is always algebraic* (in a natural way), for the local ring $\mathcal{O}_{x,V}$ of any point x of V is a homomorphic image of the local ring $\mathcal{O}_{x,X}$ of this point, when x is regarded as a point of X, whence the $\mathcal{O}_{x,V}$-module $\mathcal{F}_x$ is also an $\mathcal{O}_{x,X}$-module. The converse is not always true. However, if both $\mathcal{F}$ and $\mathcal{G}$ are algebraic and if one of them is coherent then also the other is coherent.

Now let us restrict ourselves to sheaves $\mathcal{F}$ over the projective r-space X. In practice, $\mathcal{F}$ will vanish outside the irreducible projective variety V which is the object of study, and in that case we shall say that $\mathcal{F}$ *is carried by* V.

Let $\mathcal{F}$ be an algebraic sheaf over X. A basic operation considered by Serre is the one which associates with $\mathcal{F}$, and with any integer n, another algebraic sheaf $\mathcal{F}(n)$. This sheaf $\mathcal{F}(n)$ is defined as follows:

Let $y_0, y_1, \cdots, y_r$ be homogeneous coördinates in X and let U_i be the open set defined by $y_i\neq 0$. We consider the $r+1$ sheaves $F^i=\mathcal{F}_{U_i}$ (=restriction of $\mathcal{F}$ to U_i) and we patch them up together again, but by a different rule: if $x\in U_i\cap U_j$ and $f_i\in\mathcal{F}^i_x$, $f_j\in\mathcal{F}^j_x$, then we identify f_i and f_j if

$$f_i = \left(\frac{y_i}{y_j}\right)^n f_j. \tag{1}$$

(Note that (1) makes sense since y_i/y_j is a unit in $\mathcal{O}_x$ if $x\in U_i\cap U_j$.) Every element of the stalk of $\mathcal{F}(n)_x$ shall have a unique representative in $\mathcal{F}^i_x$, if $x\in U_i$, and two representatives f_i, f_j of f (if $x\in U_i\cap U_j$) shall be related by (1). A sheaf structure is then introduced in $\mathcal{F}(n)$ in a natural way, so that $\mathcal{F}(n)$ and $\mathcal{F}$ are locally isomorphic everywhere.

It follows that $\mathcal{J}(n)$ is coherent if $\mathcal{J}$ is coherent. Note that $\mathcal{J}(0)=\mathcal{J}$.

The geometric meaning of the sheaf $\mathcal{J}(n)$ becomes clear in the important special case in which $\mathcal{J}$ is a sheaf carried by V and associated with a divisor D on V (we shall assume that V is normal). Then, for $x\in V$, $\mathcal{J}_x$ is the set of rational functions f on V such that $(f)+D>0$ locally at the point x. The global sections of $\mathcal{J}$ are given by the functions f such that $(f)+D>0$ globally on V. These functions f form a finite-dinensional vector space $\Gamma(\mathcal{J}, V)$ over k, say of dimension $\rho+1$, and the set of integral cycles $(f)+D$, $f\in\Gamma(\mathcal{J}, V)$, $f\neq 0$, is the complete linear system $|D|$, which is then of dimension ρ. The sheaf $\mathcal{J}$ associated with a divisor D will be denoted by $\mathcal{L}(D)$. Note the relation

$$(2) \qquad \dim H^0(V, \mathcal{L}(D)) = 1+\dim|D|.$$

If D and D' are linearly equivalent divisors, so that $D-D'=(\alpha)$, $\alpha\in K$, then $\mathcal{L}(D)$ and $\mathcal{L}(D')$ are isomorphic sheaves, the isomorphism being obtained by multiplying the sheaf elements of $\mathcal{L}(D)$ by α.

It is now not difficult to see that *if C denotes a hyperplane section of V and C_n is any divisor linearly equivalent to nC, then $\mathcal{L}(D)(n)$ is isomorphic with $\mathcal{L}(D+C_n)$*. It is sufficient to prove this for one particular C_n, and we shall take for C_n the divisor nC^0, where C^0 is the section of V with the hyperplane $y_0=0$ (we may assume, without loss of generality, that this hyperplane does not contain the variety V). If $f\in\mathcal{L}(D)(n)_x (x\in V)$ and f_i is a representative of f in $\mathcal{L}(D)_{U_i}$ $(0\leqq i\leqq r;\ x\in U_i)$, then $f^{(n)}=f_i(y_i/y_0)^n$ depends only on f, by (1), and belongs to $\mathcal{L}(D+nC^0)_x$. It is then immediately seen that the mapping $f\rightarrow f^{(n)}$ is an isomorphism of $\mathcal{L}(D)(n)$ with $\mathcal{L}(D+nC^0)$.

The study of the complete linear systems $|D+nC|$ is fundamental in theory of linear systems on V, and the sheaf-theoretic analogue of this is the study of the sheaves $\mathcal{L}(D)(n)$.

If D is null divisor, then $\mathcal{L}(D)$ is the sheaf $\mathcal{O}_V$. It follows that $\mathcal{O}_V(n)$ is isomorphic with $\mathcal{L}(nC^0)$.

Let $R=k[z_0, z_1, \cdots, z_r]$ be the homogeneous coördinate ring of V, where $z_0, z_1, \cdots, z_r$ are therefore strictly homogeneous coördinates of the general point of V/k. Then $R=\sum_{n=0}^{+\infty} R_n$ is graded ring, and also its integral closure $I=\sum_{n=0}^{\infty} I_n$ is a graded ring (Zariski [1]); here R_n and I_n are the sets of homogeneous elements (of R and I, respectively) which are of degree n. From known theorems on normal varieties it follows that

$$(3) \qquad \Gamma(\mathcal{O}_V(n), V)\cong I_n,$$

$$(3') \qquad \Gamma(\mathcal{O}_V(n), V)\cong R_n(=I_n), \text{ if } n \text{ is large},$$

all the isomorphisms being canonical.

If V is not normal then the following can be easily proved: *there exists a greatest graded ring* $J=\sum_{n=0}^{\infty} J_n$ *between* R *and* I *such that the conductor of* R *relative to* J *contains a power of the irrelevant ideal* $(z_0, z_1, \cdots, z_n)$($J=I$ if and only if V is normal; $J_n=J\cap I_n$), *and relations* (3), (3′) *continue to hold if we replace* I *by* J.

Another representation of the sheaf $\mathcal{O}_V(n)$ is the following: *the stalk* $\mathcal{O}_V(n)_x$ $(x\in V)$ *consists of the quotients of the form*

$$f(z_0, z_1, \cdots, z_r)/g(z_0, z_1, \cdots, z_r),$$

where f *and* g *are forms of degree* $j+n$ *and* j *respectively* (j-*arbitrary*) *and where the form* g *does not vanish at the point* x. From this representation of $\mathcal{O}_V(n)$ it follows that the product of an element of $\mathcal{O}_V(n)_x$ by a form of degree s in $y_0, y_1, \cdots, y_r$ can be identified with a well-defined element of $\mathcal{O}_V(n+s)_x$.

If $\mathcal{J}$ is an arbitrary sheaf then a representation of $\mathcal{J}(n)$ is the following: $\mathcal{J}(n)$ *is isomorphic to the tensor product* $\mathcal{J}\otimes\mathcal{O}_V(n)$, i.e.,

$$\mathcal{J}(n)_x \cong \mathcal{J}_x \underset{\mathcal{O}_x}{\otimes} \mathcal{O}_x\mathcal{O}(n)_x.$$

REMARK. From this representation of $\mathcal{J}$ it follows that the product of any element of $\mathcal{J}(n)_x$ by any form of degree h in $y_0, y_1, \cdots, y_r$ can be identified with a well-defined element of $\mathcal{J}(n+h)_x$. Consequently, if S denotes the polynomial ring $k[y_0, y_1, \cdots, y_r]$ and S_h is the set of forms of degree h then we can write $S_hH^0(V, \mathcal{J}(n)) \subset H^0(V, \mathcal{J}(n+h))$. Later we shall make use of this remark.

Before going further with our summary we shall make some general remarks about the methods of proof. There are essentially two principal methods:

a. The inductive, algebro-geometric method.

b. The abstract method of cohomological algebra of functors. We shall describe briefly these two methods.

The inductive method. Let $\mathcal{J}$ be a coherent sheaf over our variety V. The operation $\mathcal{J}\to\mathcal{J}(n)$ and the extension of $\mathcal{J}$ to the whole projective space are commutative operations. Hence we may assume that $\mathcal{J}$ is defined over the whole of X. Let $L(y)$ be an arbitrary linear form in the y's. If $f\in\mathcal{J}_x$, the collection of elements $fL(y)/y_i$, where i ranges over those indices $0, 1, \cdots, r$ for which $x\in U_i$, represents an element $\phi(f)$ of $\mathcal{J}(1)_x$. Thus we get an algebraic homomorphism of $\mathcal{J}$ into $\mathcal{J}(1)$, and by Lemma 2 (§3) the kernel $\mathcal{L}$ and cokernel $\mathcal{M}$ of ϕ are both coherent. If we replace here $\mathcal{J}$ by $\mathcal{J}(n)$, then $\mathcal{J}(1)$, $\mathcal{L}$ and $\mathcal{M}$ must be replaced by $\mathcal{J}(n+1)$, $\mathcal{L}(n)$, $\mathcal{M}(n)$. Hence we have an exact sequence of the form:

$$0 \to \mathcal{L}(n) \to \mathcal{J}(n) \to \mathcal{J}(n+1) \to \mathcal{M}(n) \to 0. \tag{4}$$

The equation $L(y)=0$ defines a hyperplane Y, and Y is of course a projective space of dimension $r-1$. It is not difficult to see that the sheaves $\mathcal{L}(n)$ and $\mathcal{M}(n)$ are zero outside of Y, hence are carried by Y. Moreover it can be verified that they are algebraic, hence coherent, as sheaves with Y as base space. The sequence (4) offers therefore a general method for applying induction with respect to the dimension m of the carrier V of $\mathcal{J}$, for the sheaves $\mathcal{L}(n)$ and $\mathcal{M}(n)$ are zero outside $V\cap Y$, and $V\cap Y$ is of dimension $m-1$. If we wish to prove a theorem which contains, among its assumptions, the assumption that V satisfies a certain condition (say, that V is irreducible, or nonsingular or normal), the success of the induction will depend, among other things, on the possibility of choosing the hyperplane Y in such a manner that also $V\cap Y$ satisfies that same condition (if the condition is one of those which we have mentioned above in parenthesis, such a choice is always possible).

The method just outlined is standard in the classical theory, and many arguments involving only this method in the analytical case can be made abstract without too great difficulty. The following outline of the proofs of the two fundamental theorems given below (Theorems 5 and 6) will illustrate this method.

THEOREM 5. *If $\mathcal{J}$ is a coherent sheaf over X then the sheaf $\mathcal{J}(n)$ is semi-fine for all sufficiently large n.*

THEOREM 6. *If $\mathcal{J}$ is a coherent sheaf over X then all the cohomology groups $H^q(X, \mathcal{J})$ are finite dimensional (as vector spaces over the ground field k).*

It is easy to see that $\mathcal{J}(n)$ satisfies condition SF.1 (see §5) if n is large. In fact, we know that each of the sheaves $\mathcal{J}_{U_i}$, $i=0, 1, \cdots, r$, is semifine. Since $\mathcal{J}(n)_x$ is essentially $\mathcal{J}_x$, for all $x\in X$, it is sufficient to show that if f is any section of $\mathcal{J}$ over one of the U_i, say over U_0, then, for large n, f can be extended to a global section g of $\mathcal{J}(n)$. From Lemma 4 and Theorem 2 (§4) it follows that $(y_0/y_i)^n f$ can be considered as the restriction to $U_0\cap U_i$ of a section $f_i^{(n)}$ of $\mathcal{J}$ over U_i, provided n is sufficiently large. The desired global section g of $\mathcal{J}(n)$ is the one defined by the $r+1$ partial sections $f_i^{(n)}$, $i=0, 1, \cdots, r$.

We now show that $\mathcal{J}(n)$ also satisfies condition SF.2 if n is large, and we begin with the case $\mathcal{J}=\mathcal{O}_V^p$. In this case we have $\mathcal{L}=0$ (essentially because each $\mathcal{O}_x$ is an integral domain) and the exact sequence (4) now specializes to $0\to\mathcal{O}^p(n)\to\mathcal{O}^p(n+1)\to\mathcal{O}^p_Y(n+1)\to 0$. If we assume by induction that $\mathcal{O}^p_Y(n+1)$ has the property SF.2, we get canonical isomorphisms: $H^q(X, \mathcal{O}^p(n))\to H^q(X, \mathcal{O}^p(n+s))$ for $q>1$ and for $s>0$. Since the mapping: $H^0(X, \mathcal{O}^p(n+1))\to H^0(Y, \mathcal{O}^p_Y(n+1))$ is an epimorphism, we get a similar isomorphism also for $q=1$. If we combine with this the remark that every coherent sheaf over $X-Y$ is semifine, we can prove that $H^q(X, \mathcal{O}^p(n))=0$, i.e., $\mathcal{O}^p(n)$ has the property SF.2. We can now treat the general case. Since $\mathcal{J}(n)$ has the property SF.1 for n sufficiently large, we have an algebraic homomorphism from $\mathcal{O}^p$ onto $\mathcal{J}(n)$. Let $\mathcal{R}$ be the kernel of this homomorphism. Then for any s we have an exact sequence of the form $0\to\mathcal{R}(s)\to\mathcal{O}^p(s)\to\mathcal{J}(n+s)\to 0$, and therefore $H^q(X, \mathcal{J}(n+s))=H^{q+1}(X, \mathcal{R}(s))$ by the special case treated above. After $r-q+1$ steps we get an isomorphism between $H^q(X, \mathcal{J}(n))$ and an $(r+1)$th cohomology group of a coherent sheaf over X, provided n is sufficiently large. However this is zero by the result stated in the beginning of this section. Thus Theorem 5 is proved.

From this theorem it follows that

$$0\to\mathcal{R}(-n)\to\mathcal{O}^p(-n)\to\mathcal{J}\to 0, \tag{5}$$

for n sufficiently large, and it is therefore sufficient to prove Theorem 6 for $\mathcal{O}^p(-n)$ and $\mathcal{R}(-n)$. Moreover, since $H^0(X, \mathcal{R}(-n))$ is a subspace of $H^0(X, \mathcal{O}^p(-n))$, and since the latter is of dimension p or zero according as $r=0$ or positive, the former is of finite dimension. We shall denote by $\mathcal{G}$ either $\mathcal{R}(-n)$ or $\mathcal{O}^p(-n)$, and we shall show that $H^q(X, \mathcal{G})$ is of finite dimension for $q>0$. If we take $\mathcal{G}$ as $\mathcal{J}$ in (4), we get $0\to\mathcal{G}(s)\to\mathcal{G}(s+1)\to\mathcal{M}(s)\to 0$. If we apply an induction on r, we may assume that $H^q(Y, \mathcal{M}(s))$ is of finite dimension for all s. Then $H^q(X, \mathcal{G}(s))$ and $H^q(X, \mathcal{G}(s+1))$ have or have not finite dimensions at the same time. However, since $H^q(X, \mathcal{G}(s))=0$ for s sufficiently large, $H^q(X, \mathcal{G})$ must be of finite dimension.

The abstract method of functorial algebra. If $\mathcal{J}$ is an algebraic sheaf over the projective space X we set

$$M(\mathcal{J}) = \sum_n H_0(V, \mathcal{J}(n)), \qquad M(\mathcal{J})_n = H_0(V, \mathcal{J}(n)),$$

where n ranges over the set of all integers and where the sum is direct. By a remark made earlier in this section, $M(\mathcal{J})$ is a graded S-module. It turns out that $\mathcal{J}$ *is coherent if and only if there exists an integer* n_0 *such that* $\sum_{n=n_0}^{+\infty} M(\mathcal{J})_n$ *if a finite S-module and that the mapping* $\mathcal{J}\to\sum_{n\geqq n_0} M(\mathcal{J})_n$ *is a* (1, 1) *correspondence between coherent sheaves* *and finite S-modules, provided two such modules which differ only in their components of low degrees are identified.* Geometric theorems can then be translated into algebraic theorems concerning certain functors derived from such modules (in particular, the so-called *extension functors*), and the powerful technique of modern functorial algebra can then be used. This method, developed in full by Serre in [**3**], has not been considered in the seminar, for the lack of knowledge and time. However, by using this method, *in conjunction with the inductive method described above*, Serre was able to prove abstractly his fundamental duality theorem (see below, §8) and therefore also the equality $P_a = p_a$ conjectured by Severi (see Zariski [**4**]; cf. also §8). So it would seem that the deeper results of theory of sheaves over algebraic varieties depend on the possibility of applying this second abstract method of functor algebra.

7. The arithmetic genera. We have now enough tools to reproduce, by sheaf-theoretic methods, the algebraic theory of arithmetic genera given by Zariski in [**4**]. The method used by Serre is largely the same as the one used by Kodaira and Spencer in [**1**]. If $\mathcal{J}$ is a coherent sheaf on our m-dimensional variety V, in the projective space, then the Euler-Poincaré characteristic of V, with coefficients in $\mathcal{J}$, is defined by

$$\chi(V, \mathcal{J}) = \sum_{q=0}^{m} (-1)^q \dim H^q(V, \mathcal{J}).$$

(Recall Theorem 6 and note that $\dim H^q(V, \mathcal{J}) = 0$ if $q > m$).

If we apply an induction using the fundamental exact sequence (4) of §6, we can show easily that $\chi(V, \mathcal{J}(n))$ is a polynomial in n for all n, of degree at most equal to m. Also, for n sufficiently large, this polynomial coincides with $\dim H^0(V, \mathcal{J}(n))$, by the property SF.2 of $\mathcal{J}(n)$ (Theorem 5). In particular, if we remark that, for large n, $H^0(V, \mathcal{O}(n))$ is canonically isomorphic with the module R_n of homogeneous elements of degree n in the homogeneous coördinates ring R of V (see §6; also Zariski [**1**]), we see that $\chi(V, \mathcal{O}(n))$ is nothing but the Hilbert polynomial of V. Therefore

$$\chi(V, \mathcal{O}) = \sum_{q=0}^{m} (-1)^q \dim H^q(V, \mathcal{O})$$

is the constant term of this polynomial, and consequently

$$\chi(V, \mathcal{O}) = 1 + (-1)^m p_a, \tag{1}$$

where p_a is the arithmetic genus of V. Since the sheaf theory is invariant under biregular transformations, $\chi(V, \mathcal{O})$ is a biregular invariant of V. This theorem was proved by Muhly and Zariski [1] for normal varieties, but under more general transformations. On the other hand, Kodaira and Spencer [1] used the above expression of the arithmetic genus to prove its *birational* invariance for nonsingular varieties over the complex field. Firstly, $H^q(V, \mathcal{O})$ is isomorphic to the space of "harmonic forms of type $(0, q)$," by Dolbeault's isomorphism and then by the decomposition formula of harmonic forms. Secondly, if we pass to the complex conjugates of those forms, we get harmonic forms of type $(q, 0)$, i.e., algebraic differential forms of degree q of the first kind. Finally, these forms behave nicely under birational transformations, and, e.g., the number g_q of linearly independent forms is birationally invariant for every q. Done!

It is worthwhile to remark here the following: Kodaira and Spencer considered only analytic sheaves over nonsingular varieties with the usual Hausdorff topology. Thus, in the classical case, there are two cohomology groups, one is analytic, and the other is algebraic. Also, theorems concerning these cohomology groups are similar. Since the Zariski topology is weaker than the Hausdorff topology, we can define a mapping from the algebraic cohomology groups into the corresponding analytic cohomology groups which commutes with homomorphisms in both theories. However, since the mapping is an isomorphism for zero-dimensional cohomology groups and since the $\mathcal{J}(n)$ have the property SF.2 for coherent sheaves $\mathcal{J}$ in both theories, the mapping gives an "isomorphism" of the two theories.

Let us now assume that V is nonsingular. For every divisor D on V we define a numerical character $\chi_V(D)$ of D by

$$\chi_V(D) = \chi(V, \mathcal{O}) - \chi(V, \mathcal{L}(-D)). \tag{2}$$

Then $\chi_V(D)$ has the characteristic properties which Zariski [4] proved for the "virtual arithmetic genus $p_a(D)$ of D with respect to V," and from this it follows easily that

$$\chi_V(D) = 1 + (-1)^{m-1}p_a(D). \tag{3}$$

This we can show by taking $L(D)$ as F in the fundamental exact sequence (4) of §6 (cf. Kodaira-Spencer [1]). Thus the virtual arithmetic genera are connected with sheaf theory.

8. **Theorem of Riemann-Roch and related topics.** If D is a divisor on V, we shall write $h^q(D)$ for $\dim H^q(V, \mathcal{L}(D))$ and $\chi(V, D)$ for $\chi(V, \mathcal{L}(D))$. We shall also write $\chi(V)$ for $\chi(V, \mathcal{O})$. If C denotes a hyperplane section of V, then it is known (Zariski [4]) that

$$1 + \dim |D + nC| = (-1)^m\{p_a(V) + p_a(-D - nC)\}, \tag{1}$$

for n large. This result can also be derived, by sheaf-theory, from (1) and (3) of §7 and from the fact that the sheaf $\mathcal{L}(D+nC)$ $(=\mathcal{L}(D)(n))$ is semifine if n is large (Theorem 5, §6). Namely, if n is large, then the left-hand side of (1) coincides with $\chi(V, D+nC)$, while the right-hand side is equal to $\chi(V)-\chi_V(-D-nC)$, and (1) now follows from the definition of the symbol $\chi_V(-D-nC)$ (see (2), §7).

The expression $(-1)^m\{p_a(V)+p_a(-D)\}+1$, which is not in general equal to the dimension of the complete linear system $|D|$, may be said—to use a time-honored expression of the Italian geometers—to represent the *virtual dimension of* $|D|$. Relation (1) tells us that for large n the virtual dimension of $|D+nC|$ coincides with the effective dimension. We have, by (1) and (3) of §7:

$$(-1)^m\{p_a(V) + p_a(-D)\} = \chi(V, D). \tag{2}$$

This precise equality can be regarded, in a limited sense, as *a generalization of the Riemann-Roch theorem to varieties*. Sheaf theory tells us that *the virtual dimension of* $|D|$, *increased by* 1, *equals the Euler-Poincaré characteristic of* V, *with coefficients in the sheaf* $L(D)$.

In this connection it may be pointed out that the following formula is implicitly contained in Zariski [**4**]:

$$(-1)^m\{p_a(V) + p_a(-D)\} = \sum_{q=0}^{m} (-1)^q s_q(D), \tag{2'}$$

where $s_0(D)=1+\dim |D|$;

$s_1(D)=$deficiency of the linear system $\{D^{(1)}\}$ cut out on a generic C_n by the complete linear system $|D+C_n|$; here n is a sufficiently high integer, and C_n is the section of V with a hypersurface of order n.

$s_2(D)=$deficiency of the linear system $\{D^{(2)}\}$ cut out on $C_n\cdot C'_{n'}$ by the complete linear system $|D^{(1)}|$; here n' is sufficiently large with respect to n, and $C_{n'}'$ is generic with respect to C_n; and so on, except that $s_m(D)$ is defined as the index of specialty of the zero-dimensional cycle $D^{(m-1)}$ cut out on the curve $C_n\cdot C'_{n'} \cdots C_{n(m-2)}^{(m-2)}$ by $C_{n(m-1)}^{(m-1)}$. It is clear that $s_0(D)=h^0(D)$. Using the duality relation (8) established below it is possible to show that $s_m(D)=h^m(D)$.

In the case of curves $(m=1)$, $p_a(V)$ is the genus of the curve V, and $p_a(-D)=-\deg D-1$, and thus we have by (2):

$$\dim |D| = \deg D - g + h^1(D).$$

It follows by the classical Riemann-Roch theorem that

$$h^1(D) = h^0(K - D), \tag{3}$$

where K is a canonical divisor on V.

For $m>1$ a satisfactory algebro-geometric characterization of the cohomological dimension $h^q(D)$, $q>0$, is still missing, except for $q=m$, in which case we have a fundamental duality relation similar to (3):

$$h^m(D) = h^0(K - D), \tag{4}$$

where K is again a canonical divisor on V, i.e., the divisor of an m-fold differential on V. This equality follows readily from the so-called "lemma of Enriques-Severi-Zariski" proved by Zariski [4], which states that *"if D is any divisor on V and C_n is a general section of V by a hypersurface of order n, then for n sufficiently large the linear system* $\mathrm{Tr}_{C_n}|D|$, *cut out on C_n by the complete linear system $|D|$, is itself complete."* An equivalent formulation is the following equality:

$$\dim |D| = \dim |D\cdot C_n|, \tag{5}$$

if n is large, since for large n we have $\dim |D| = \dim \mathrm{Tr}_{C_n}|D|$. If C_n' is another member of $|C_n|$, different from C_n, then—always under the assumption that C_n is a general section of V—it can be shown that the following sequence is exact:

$$0 \to \mathcal{L}(D) \xrightarrow{i} \mathcal{L}(D + C_n') \xrightarrow{j} \mathcal{L}(C_n, (D + C_n')\cdot C_n) \to 0$$

(where the quotient sheaf $\mathcal{L}(C_n, (D+C_n')\cdot C_n)$ denotes the sheaf *on* C_n, defined by the divisor $(D+C_n')\cdot C_n$), and the corresponding exact cohomology sequence

$$\begin{aligned} \cdots &\to H^q(V, \mathcal{L}(D + C_n')) \xrightarrow{j^*} H^q(C_n, \mathcal{L}((D + C_n')\cdot C_n)) \\ &\xrightarrow{\delta^*} H^{q+1}(V, \mathcal{L}(D)) \xrightarrow{i^*} H^{q+1}(V, \mathcal{L}(D + C_n')) \to \cdots \end{aligned} \tag{6}$$

shows that if $q>0$ and n is sufficiently large, then the two middle terms are isomorphic (since $\mathcal{L}(D+C_n')$ is then semifine) and hence

$$h^q(C_n, (D + C_n')\cdot C_n) = h^{q+1}(D), \tag{7}$$

(n—large, $q>0$). In view of (3), we can prove (4) by induction with respect to m: Writing (7) for $q=m-1$ (we may assume $m>1$) and using our induction hypothesis we find

$$h^m(D) = h^0(C_n, \overline{K} - (D + C_n')\cdot C_n),$$

where $\overline{K}$ is a canonical divisor on C_n. It is known (Zariski [4]) that $\overline{K} \equiv (K+C_n')\cdot C_n$. Hence $h^m(D) = h^0(C_n, (K-D)\cdot C_n)$, i.e.,

$$h^m(D) = 1 + \dim |(K - D)\cdot C_n|, \tag{8}$$

n large, and now (4) follows in view of the lemma of Enriques-Severi-Zariski (5), as applied to the divisor $K-D$.

Conversely, the Zariski result (5) follows from the duality formula (4), in view of (8).

For $m=2$ the Riemann-Roch formula (2) together with the duality formula (4) yields the relation

$$\dim |D| = p_a(V) + p_a(-D) - \dim |K - D| + h^1(D),$$

and hence the well-known *Riemann-Roch inequality* $\dim |D| \geqq p_a(V) + p_a(-D) - \dim |K-D|$, and $h^1(D)$ therefore coincides with the so-called *superabundance* of $|D|$.

Another result that is equivalent with the lemma of Enriques-Severi-Zariski is the following:

$$h^1(D - C_n) = 0, \tag{9}$$

if n is large. In the proof of (9) we may replace D by $D+C_h$, h arbitrary. Therefore we may assume that $h^1(D)=0$. If in (6) we replace D by $D-C_n$, we find, for large n:

$$\begin{aligned} 0 \to H^0(V, \mathcal{L}(D - C_n)) \xrightarrow{i^*} H^0(V, \mathcal{L}(D)) \xrightarrow{j^*} H^0(C_n, \mathcal{L}(D\cdot C_n)) \\ \to H^1(V, \mathcal{L}(D - C_n)) \to 0 \to \cdots . \end{aligned} \tag{10}$$

Now, Zariski's result (5) signifies that j^* is an epimorphism if n is large. Since (10) is exact, we must have $H^1(V, \mathcal{L}(D-C_n))=0$, i.e., $h^1(D-C_n)=0$. Conversely, if $h^1(D-C_n)=0$, then (10) shows that j^* is an epimorphism.

We have thus three basic but equivalent statements: (4), (5) and (9). We have outlined the proofs of (4) and (9) by using the relation (5) which we have proved in [4] by a direct algebro-geometric argument (and for normal, not only nonsingular, varieties). Serre gave two sheaf-theoretic proofs of these results. The first proof (unpublished; cf. Serre [4]) uses explicitly the m-fold differentials on V and is—in part—an adaptation of an argument used by Weil [1] in the one-dimensional case. This proof is applicable only to nonsingular varieties. The second proof (Serre [3]) uses the algebra of functors, goes much further, and is applicable also to normal varieties. The first proof was discussed in detail in the seminar and will now be briefly outlined. After that, we shall say a few words about the second proof.

We denote by $\Omega^p(D)$ the sheaf of germs of p-fold differentials on V

which are (locally) multiples of $-D$. This is a coherent sheaf, since it is locally isomorphic with the $\binom{m}{p}$-fold direct product of the sheaf $\mathcal{O}$. Note that $\Omega^0(D)=\mathcal{L}(D)$. We write Ω^p for $\Omega^p(0)$: this is the sheaf of germs of holomorphic differentials.

First of all one proves that $H^m(V, \Omega^m)$ is of dimension 1 over k. For $m=1$ this is a consequence of the residue theorem for abelian differentials on an algebraic curve V ("there exists a differential with preassigned poles of order $\leqq 1$ and preassigned residues at these poles, provided the sum of residues is zero"). The general case can be reduced to the case $m=1$ by the inductive procedure described in §6. The fundamental exact sequence (4) of §6 to be used in this case is the one in which $\mathcal{J}$ is replaced by Ω^m and C is replaced by C_n, n large. A similar argument shows that $H^m(V, \Omega^m(D))$ is zero for any strictly positive divisor D. These two assertions can be regarded as generalizations of the "residue theorem" for abelian differentials.

Now, if ω is any element of $H^0(V, \Omega^m(-D))$ (i.e., ω is an m-fold differential which is globally a multiple of D), then the multiplication by ω of each stalk of $\mathcal{L}(D)$ defines an element of Hom $[\mathcal{L}(D), \Omega^m]$. Therefore ω also defines an element $J_D(\omega)$ of Hom_k $[H^m(V, \mathcal{L}(D)), H^m(V, \Omega^m)]$, and since $H^m(V, \Omega^m)\cong k$, $J_D(\omega)$ is an element of the dual space of $H^m(V, \mathcal{L}(D))$. We thus have *a mapping J_D of $H^0(V, \Omega^m(-D))$ into the dual space of $H^m(V, \mathcal{L}(D))$*. We shall denote this dual space by $W(-D)$. Since $H^0(V, \Omega^m(-D))$ is obviously isomorphic with $H^0(V, K-D)$, the duality relation (4) will follow if it is proved *that J_D is an isomorphism*.

If D is a multiple of D', the cokernel of the inclusion map i: $\mathcal{L}(D')\to\mathcal{L}(D)$ is carried by a variety of dimension $m-1$. Therefore i^*: $H^m(V, \mathcal{L}(D'))\to H^m(V, \mathcal{L}(D))$ is an epimorphism, hence the dual of i^* is a monomorphism as a mapping of $W(-D)$ into $W(-D')$. We can consider the inductive limit W of $W(-D)$ by these maps. It is easy to introduce in W a structure of a vector space over the function-field K of V such that the inductive limit J of J_D is a K-linear mapping of the space of all global m-forms on V into W. Here,—*and this is the key point of the proof—if $J(\omega)$ is contained in $W(-D)$, then ω is necessarily contained in $H^0(V, \Omega^m(-D))$*. The proof is indirect and is as follows:

If we assume that ω is not contained in $H^0(V, \Omega^m(-D))$, the reduced expression of $(\omega)-D$ is of the form C_1-C_2 where C_1 is nonnegative and C_2 is strictly positive. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be cokernels of the inclusion maps $\mathcal{L}((\omega)-C_1)\to\mathcal{L}(D)$ and $\mathcal{L}((\omega))\to\mathcal{L}((\omega)+C_2)$, respectively. Then we get the following commutative diagram of exact sequences:

$$\begin{array}{l}\to H^{m-1}(V, \mathcal{M}_1) \to H^m(V, \mathcal{L}((\omega) - C_1)) \to H^m(V, \mathcal{L}(D)) \to 0 \\ \qquad\quad \downarrow \\ \to H^{m-1}(V, \mathcal{M}_2) \to H^m(V, \mathcal{L}((\omega))) \to 0 \\ \qquad\quad \downarrow \\ \qquad\quad 0\end{array}$$

We note that $J_{(\omega)}(\omega)$: $H^m(V, \mathcal{L}(\omega)) \to H^m(V, \Omega^m)$ is an isomorphism. Moreover, since $J(\omega)$ is contained in $W(-D)$, by definition, $J_{(\omega)-C_1}(\omega)$, i.e., the product of $H^m(V, \mathcal{L}((\omega)-C_1)) \to H^m(V, \mathcal{L}((\omega)))$ and $J_{(\omega)}(\omega)$ vanishes on the image of $H^{m-1}(V, \mathcal{M}_1)$ in $H^m(V, \mathcal{L}((\omega)-C_1))$. However the commutativity and the exactness of the diagram show the opposite of this assertion, and this gives us the desired contradiction.

As a simple consequence, we see that J is an isomorphism. On the other hand, *W is of dimension one over K*. In fact, let C be a general hyperplane section of V. Then, by an induction on m, we can show that ord $(V)\cdot n^m/m!$ is the dominant term of dim $H^m(V, \mathcal{L}(D-nC))$ as a function of n, this function being equal to dim $W(-D+nC)$. Since ord $(V)\cdot n^m/m!$ is the highest term of dim $|nC|$ as a polynomial in n for n sufficiently large, we can readily get a contradiction if W is not of dimension one over $F(V)$. Since J is a monomorphism, it is now shown that J is an isomorphism. If we apply the key result which we have established earlier, we conclude that J_D is an isomorphism. Thus (4) is proved.

In [**3**] Serre proves, by using extension functors, the following fundamental result:

THEOREM 7. *Let $\mathcal{J}$ be a coherent sheaf over the projective r-space X and let p be an integer $\geqq 0$. In order that $H^q(X, \mathcal{J}(-n))$ be zero for large n and for $0 \leqq q < p$ it is necessary and sufficient that for every $x \in X$ the $\mathcal{O}_x$-module $\mathcal{J}_x$ have cohomological dimension $\leqq r-p$ (i.e., that there exist an exact sequence $0 \to L_{r-p} \to L_{r-p-1} \to \cdots \to L_0 \to \mathcal{J}_x \to 0$, where each L_i is a free $\mathcal{O}_x$-module; here $\mathcal{O}_x$ denotes the local ring of x).*

Now, suppose that $\mathcal{J}$ is zero outside of our m-dimensional variety V (i.e., that $\mathcal{J}_x = 0$ for all $x \notin V$). For $x \in V$ let $\mathfrak{o}_x$ be the local ring of x on *the variety V. Assume furthermore that $\mathcal{J}_x$ is a free module over $\mathfrak{o}_x$.* If V is nonsingular then it is known that $\mathfrak{o}_x$ has cohomological dimension $r-m$ over $\mathcal{O}_x$, $x \in V$. If V is normal, then it can be shown that for all $x \in V$ the $\mathcal{O}_x$-module $\mathfrak{o}_x$ has dimension $\leqq r-2$ (Serre [**3**]). Hence by Theorem 7 we find that for large n

(11) $\qquad H^q(V, \mathcal{J}(-^n)) = 0, \qquad 0 \leqq q \leqq m-1$, if V is nonsingular.

and

(12) $$H^q(V, \mathcal{J}(-n)) = 0, \qquad 0 \leqq q \leqq 1, \text{ if } V \text{ is normal.}$$

Relation (12), for $q=1$, includes the lemma of Enriques-Severi-Zariski as a special case. If V is nonsingular, every sheaf $\mathcal{L}(D)$ defined by a divisor D is free over $\mathfrak{o}_x$ for every x in V. Hence, we have by (11):

(13) $$h^q(D - C_n) = 0, \qquad 0 \leqq q \leqq m - 1, n\text{—large.}$$

For $q=1$ we find relation (9). Essentially the same proof which enabled us to derive from (9) the duality relation (4) enables one to derive from (13) the duality relations

(14) $$h^q(D) = h^{m-q}(K - D), \qquad 0 \leqq q \leqq m.$$

The following is a noteworthy application of (14). We have, by (14): $\chi(V,D) = (-1)^m\chi(V,K-D)$. For $D=0$, we find $\chi(V) = (-1)^m\chi(V, K)$, and this yields, by (2), §7:

(15) $$\chi_V(-K) = \begin{cases} 2\chi(V) & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Relation (15) is equivalent with the equality $P_a = p_a$ (Zariski [4]), and thus we have a proof of Severi's conjecture $P_a = p_a$.

In conclusion we mention the following unsolved problem: do the symmetry relations $h^{p,q} = h^{q,p}$ hold? Here $h^{p,q} = \dim H^p(V, \Omega^q)$. It is known that these equalities hold in the complex domain, but the proof depends on properties of harmonic integrals and cannot be therefore algebraicized.

As was pointed in §7, the symmetry relations in the classical case, and in particular the relations $h^{p,0} = h^{0,p}$, yield the following expression of the arithmetic genus: $p_a = h^{m,0} - h^{m-1,0} + \cdots + (-1)^{m-1}h^{1,0}$, where $h^{p,0}$ is the number of linearly independent p-fold differentials of the first kind on V. In the case of surfaces this leads to the equality $p_g - p_a = h^{1,0}$, where $p_g = h^{2,0}$ is the geometric genus of the surface. In the abstract case it is highly probable that the above expression for the genus may still be valid even if the symmetry relations turned out to be false.

BIBLIOGRAPHY

S. BOCHNER

1. *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. vol. 52 (1946).

H. CARTAN

1. *Sur les matrices holomorphes de n variables complexes*, J. Math. Pures Appl. vol. 19 (1940).
2. *Idéaux de fonctions analytiques de n variables complexes*, Ann. École Norm. vol. 61 (1944).
3. Seminar, 1950–1951, 1951–1952, 1953–1954.

H. CARTAN-J-P. SERRE
1. *Un théorème de finitude concernant les variétés analytiques compactes*, C. R. Acad. Sci. Paris vol. 237 (1953).

P. DOLBEAULT
1. *Sur la cohomologie des variétés analytiques complexes*, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 175–177.

F. HIRZEBRUCH
1. *Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954).

K. KODAIRA
1. *On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953).
2. *On Kähler varieties of restricted type*, Ann. of Math. vol. 60 (1954).

K. KODAIRA-D. C. SPENCER
1. *On arithmetic genera of algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953).
2. *Divisor class groups on algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953).
3. *On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953).

J. LERAY
1. *L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue*, J. Math. vol. 29 (1950).

H. T. MUHLY-O. ZARISKI
1. *Hilbert's characteristic function and the arithmetic genus of an algebraic variety*, Trans. Amer. Math. Soc. vol. 69 (1950).

K. OKA
1. *Sur les fonctions analytiques de plusieurs variables* VII, Bull. Soc. Math. France vol. 78 (1950).

J-P. SERRE
1. *Faisceaux analytiques cohérents sur une variété algébrique*, written in the fall, 1953. (Cf. Sém. Bourbaki, mars 1954).
2. *Faisceaux algébriques*, written in the spring, 1954.
3. *Faisceaux algébriques cohérents*, Ann. of Math. vol. 61 (1955).
4. *Cohomologie et géométrie algébrique*, written in the fall, 1954. (Cf. Proc. Amsterdam Congress).

A. WEIL
1. *Zur algebraischen Theorie der algebraischen Funktionen*, J. Reine Angew. Math. vol. 175 (1938).
2. *Sur les théorèmes de de Rham*, Comment. Math. Helv. vol. 26 (1952).

O. ZARISKI
1. *Some results in the arithmetic theory of algebraic varieties*, Amer. J. Math. vol. 61 (1939).
2. *The compactness of the Riemann manifold of an abstract field of algebraic functions*, Bull. Amer. Math. Soc. vol. 50 (1944).
3. *Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields*, Memoirs of the American Mathematical Society, no. 5, 1951.
4. *Complete linear systems on normal varieties and a generalized lemma of Enriques-Severi*, Ann. of Math. vol. 55 (1952).

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