THE MATHEMATICAL THEORY OF THREE-DIMENSIONAL CAVITIES AND JETS

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1. Introduction. The theory of two-dimensional steady flows with free streamlines has been studied intensively for over a century and a thorough knowledge has been obtained both of methods for solving special problems in closed form and of techniques for establishing the existence of solutions for more general models. In contrast, the principal investigations concerning three-dimensional free surface flow have only appeared during the past decade and progress in this direction has been confined almost entirely to problems possessing axial symmetry. Even in the case of axial symmetry, most of the discussions have centered about existence theorems [2] or qualitative properties of the flow [3; 4], and very little has been learned about explicit treatment of the specific models which arise in engineering applications. We shall describe in the present article a systematic procedure for the numerical solution of the free boundary problem which was developed in an effort to fill this gap.

The chief significance of the method which we shall outline lies in the wide variety of free boundary problems to which it applies. However, for the sake of clear, concise presentation we prefer to restrict our attention to one quite special, albeit important, example, with the belief that in this manner we illustrate the basic ideas better and still indicate their full scope. Thus we wish to place as much emphasis on the general approach to be described as we give to the particular numerical results mentioned, some of which might even be obtained more simply by a slightly different attack [1].

We shall study in detail the axially symmetric Riabouchinsky flow past a circular disk. This model consists of a flow parallel to the x-axis past an obstacle composed of two identical disks, perpendicular to the x-axis, which are joined by a free surface attached at their perimeters. The motion is governed in the meridian (x, y)-plane by a Stokes stream function ψ satisfying the partial differential equation

$$\psi_{xx} + \psi_{yy} - \frac{1}{y}\psi_y = 0. \tag{1}$$

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Along the x-axis, or axis of symmetry, along the two circular disks $x = \pm X$, $0 \leqq y \leqq Y$ comprising the nose and tail of the obstacle, and along the free streamline joining these disks in the meridian plane, the Stokes stream function ψ vanishes,

$$\psi = 0. \tag{2}$$

In addition, along the free streamline the normal derivative $\partial\psi/\partial n$ satisfies the condition

$$\frac{1}{y}\frac{\partial\psi}{\partial n} = 1, \tag{3}$$

which states that the pressure is constant there. At infinity ψ has the expansion

$$\psi = \frac{y^2}{2(1+\sigma)^{1/2}} - \frac{ay^2}{(1+\sigma)^{1/2}r^3} + \cdots, \qquad r^2 = x^2 + y^2, \tag{4}$$

where σ is the so-called cavitation parameter of the flow and where the coefficient a is connected with the virtual mass of the obstacle [1; 2].

In the physical interpretation of the Riabouchinsky model, the free surface is to be thought of as a surface of discontinuity separating the moving liquid from a vapor cavity which is considered to be at rest. The principal difficulty encountered in the mathematical analysis of the model lies in finding the shape of the free streamline, which is presumably determined by the extra boundary condition (3). The cavitation parameter σ, which gives a dimensionless measure of the difference between the pressure at infinity and the pressure in the vapor cavity, varies monotonically with the distance $2X$ between the two disks generating the obstacle, for a fixed choice of their radius Y, and hence its determines the size of the cavity.

We shall be interested in developing a systematic, and at the same time practical, procedure for calculating the stream function ψ characterized by the conditions (1), (2), (3) and (4). The first possibility which suggests itself is to attempt to derive a construction of ψ from the proof of the existence of the solution of the axially symmetric free surface flow problem [2]. The existence proof is based on a variational principle which states that the flow with a free boundary fulfilling (3) is the one which solves the extremal problem

$$4\pi a - (1+\sigma)V = \text{minimum} \tag{5}$$

among all flows ψ of the above type satisfying (1), (2) and (4) alone, where a and σ are defined by (4) and where V denotes the volume of the varying obstacle. This minimum principle does not actually lead to a useful construction of the flow pattern, but merely serves to indicate that the stationary functional on the left in (5), which represents the energy of the motion, can be estimated from above with exceptional accuracy by computing the corresponding quantity for the flow past an obstacle whose boundary approximates the true free surface.

Although the energy is in general a quantity of secondary physical interest in hydrodynamics, it turns out that for our special example of the steady state cavity flow past a circular disk we can express the drag in terms of $4\pi a-(1+\sigma)V$ and thus can apply the result (5) effectively. We define the drag coefficient $C_D=C_D(\sigma)$ by the formula

$$\frac{C_D}{1+\sigma} = 1 - \frac{2}{Y^2}\int_0^Y \left(\frac{\partial\psi}{\partial n}\right)^2 \frac{dy}{y}, \tag{6}$$

where the integral is evaluated over only one of the circular disks forming the obstacle. The dimensionless coefficient C_D is a measure of the pressure force acting on this disk, which is to be thought of as the actual body placed in the fluid, whereas the remaining disk should be interpreted as an idealization of the turbulent wake at the opposite end of the cavity. The relation between the drag and the energy for flow past a circular disk can be written in the form

$$\frac{2\pi XY^2}{3} C_D = (1+\sigma)V - 4\pi a, \tag{7}$$

and it obviously permits us to obtain from the variational principle (5) good lower bounds on the drag coefficient C_D whenever the cavitation parameter σ is known in terms of X and Y.

Contour integration yields an easy proof of the identity (7). We note that, together with ψ, the function $\phi=x\psi_x+y\psi_y$ is also a solution of (1). Hence, letting s denote arc length, we have

$$\int \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)\frac{ds}{y} = 0 \tag{8}$$

whenever the path of integration is a simple closed curve surrounding a portion of the flow region. If we choose for this path a large semicircle and two segments of the x-axis, plus the intersection of the meridian plane with the two disks and the free surface bounding our obstacle, and if we allow the radius of the large semicircle to become infinite, we find readily that (8) leads to (7).

The relation (7) provides only one step toward a description of the cavity flow past a disk. A disadvantage of the formula is that when

X and Y are given, not only the coefficient a, but also the cavitation parameter σ must be calculated before an application can be made. Even after this has been done, a loss of significant figures can result from the cancellation on the right between the relatively large terms $(1+\sigma)V$ and $4\pi a$. Thus we turn in the next section to a completely independent, and in practice more effective, attack on the free boundary problem which, however, has not yet produced a rigorous proof of the existence of the solution.

2. **Stationary combination of the boundary conditions.** As we pointed out earlier, the main trouble in solving the boundary value problem (1), (2), (3), (4) is our ignorance of the position of the free surface. In order to overcome this difficulty, we find a linear combination of the two boundary conditions (2) and (3) which is stationary with respect to normal displacements of the free streamline. This enables us to derive a decisively improved approximation to the free boundary from any reasonably chosen first guess and thus to set up a rapidly convergent iterative scheme for the solution of the problem.

On the free streamline we can rewrite the differential equation (1) for ψ in terms of normal and tangential coordinates in the form

$$\frac{\partial}{\partial n}\frac{1}{y}\frac{\partial \psi}{\partial n}+\frac{\kappa}{y}\frac{\partial \psi}{\partial n}=0 \tag{9}$$

by virtue of the boundary condition (2), where κ denotes the curvature of the streamline. Through another application of the boundary condition (2), we derive from (9) the result

$$\frac{\partial}{\partial n}\left(\frac{1}{y}\frac{\partial \psi}{\partial n}+\frac{\kappa}{y}\psi\right)=0. \tag{10}$$

We can combine (2) and (3) on the free streamline to obtain

$$\frac{1}{y}\frac{\partial \psi}{\partial n}+\frac{\kappa}{y}\psi=1 \tag{11}$$

there, and (10) shows that this particular boundary condition is stationary in the sense that a shift of the free boundary by an infinitesimal amount δn along its inner normal results in an error of the order of magnitude $(\delta n)^2$ in the relation (11), with the relevant quantities evaluated along the displaced curve. Thus the solution ψ of the boundary value problem defined by the partial differential equation (1), by the boundary condition (2) imposed along the x-axis and on the circular disks bounding our obstacle, by the asymptotic expansion (4) at infinity, and by the boundary condition (11) imposed along an approximation to the free streamline corresponding to a normal shift of the true curve by the amount δn, yields an approximation for the stream function of the actual free surface flow with an error of the order of magnitude $(\delta n)^2$.

On our first approximation to the free boundary, the stream function ψ found in the above manner will not in general vanish, and, indeed, the quantity

$$\delta n = -\psi/y \tag{12}$$

is easily seen to represent a normal shift that carries this first guess into a second approximate free streamline which is accurate up to terms of the order $(\delta n)^2$. We can repeat the construction of ψ, using this improved form of the free surface, to obtain a third approximation with an error of the order $(\delta n)^4$, and so forth. Thus an iterative procedure, based essentially on linearization, is established for the solution of the free boundary problem. The convergence of the method is quite rapid, since the error is virtually squared at each successive step.

It should be pointed out that in mathematical examples the geometric parameters X and Y will be given rather than the cavitation parameter σ, which appears in the expansion (4). No real difficulty is involved here, since it is known [1] that the stream function ψ has at the point of separation (X, Y) of the free streamline from the fixed boundary a regular series expansion in terms of the real and imaginary parts of the complex variable $(x+iy-X-iY)^{1/2}$, and since the boundedness of the velocity of the flow thus implies the nontrivial requirement

$$\lim_{x\to X, y\to Y} (x + iy - X - iY)^{-1/2}\psi(x, y) = 0 \tag{13}$$

there, which suffices to determine σ uniquely. In practice it is best to sharpen (13) by adding the more precise condition

$$\frac{1}{y}\frac{\partial\psi}{\partial n} = 1 \tag{14}$$

at the point of separation.

The boundary value problem defined by (1), (4), (11), (13) and (14), plus (2) along the fixed boundary, has a unique solution, and we have found in specific applications that it overcomes quite effectively the difficulty arising because the shape of the free boundary is not given in advance. Thus there only remains to discuss a linear mixed boundary value problem in a specific domain. The following section will be devoted to a numerical treatment of this in principle simpler question.

3. Numerical analysis of the flow pattern. The method we shall use to solve numerically the boundary value problem (1), (4), (11), (13), (14), with (2) prescribed along the fixed boundary, consists in expressing the stream function ψ as a linear combination

$$\psi = \sum_{j=1}^{N} \lambda_j p_j \tag{15}$$

of suitable known solutions p_j of (1) with coefficients λ_j determined by the requirement that the boundary conditions be satisfied in the sense of a least squares approximation at $M > N$ properly selected points. We thus choose the simplest, most direct type of interpolation to fit the boundary conditions in preference, for example, over that based on the Dirichlet integral associated with the partial differential equation (1), since we can thus avoid numerical integration and the resulting difficulties with accuracy.

We reject two other less promising approaches for the solution of our boundary value problem, namely, the method of integral equations and the method of finite differences. It would be quite tedious to calculate with sufficient accuracy the integrals involved in the first of these two methods because of the logarithmic singularity of the kernel associated with the differential equation (1), and thus the computational work would considerably exceed what we are prepared to undertake here. The same criticism of prohibitively onerous calculation applies even more strongly to the method of finite differences, and it is questionable whether relaxation should be attempted at all in cases, such as the one here, where many types of explicit solutions of the differential equation are known. The principal disadvantage of the interpolation technique based on (15) which we intend to use is that its accuracy is worst at the boundary, but this drawback is offset by the rapid over-all convergence of the procedure. It should be noted, furthermore, that the parameters a and σ appearing in the formula (7) for the drag coefficient C_D can be computed exceptionally well by our method.

For a first guess of the shape of the free boundary we choose an affine transformation of the free streamline of the relevant plane Riabouchinsky flow past a flat plate. This curve can be described completely in terms of the Jacobian elliptic functions sn t, cn t, dn t and zn t, whose modulus we denote by k. Thus our first approximation of the free boundary has the form

$$x = k^{-1} \operatorname{zn} t + tk^{-1}(EK^{-1} - k^{*2}), \tag{16}$$

$$y = k^{-1}(k^{*2} + E^* - k^2K^*) + \beta(\operatorname{dn} t - k^*), \tag{17}$$

where k^* is the complementary modulus such that $k^{*2}+k^2=1$, where K and E are the complete elliptic integrals of the first and second kinds with modulus k, where K^* and E^* are the complete elliptic integrals of the first and second kinds with modulus k^*, and where β is a parameter associated with the above affine mapping. The variable t ranges over the interval $-K \leqq t \leqq K$, and the coordinates X and Y of the point of separation are given by

$$X = k^{-1}(E - k^{*2}K), \qquad Y = k^{-1}(k^{*2} + E^* - k^2K^*). \tag{18}$$

For our numerical example we assign to X and Y specific values by setting $k=.96$. Furthermore, we take $\beta=.25629688$ on the basis of earlier calculations [1] which are equivalent, in effect, to one iteration of the scheme outlined in the previous section. The second iteration, to be discussed here, will yield an adequate description of the physical phenomena connected with the free surface flow problem.

To define solutions of (1) for substitution into (15) which are appropriate for a representation of the flow near the separation point, we shall need the two pairs of elliptic coordinates μ_m and ν_m given by

$$\begin{aligned}\mu_m = [\{X + (-1)^m x\}^2 + y^2 - Y^2 + ([\{X + (-1)^m x\}^2 \\ + y^2 - Y^2]^2 + 4Y^2\{X + (-1)^m x\}^2)^{1/2}]^{1/2}/(2^{1/2}Y),\end{aligned} \tag{19}$$

$$\nu_m = \frac{X + (-1)^m x}{Y\mu_m}, \tag{20}$$

for $m=1, 2$. The coordinate systems μ_1, ν_1 and μ_2, ν_2 have a square root behavior which is suitable for expansion of the stream function ψ in neighborhoods of the points (X, Y) and $(-X, Y)$. However, it is necessary to exercise a certain amount of caution if we wish to compute, for example, μ_1 and ν_1 in terms of t near $t=K$. The proper technique is to use the expressions

$$x - X = k^{-1}(t - K)(EK^{-1} - k^{*2}) + k^{-1} \operatorname{zn} t, \tag{21}$$

$$y^2 - Y^2 = 2Y\beta k^2 \frac{\operatorname{cn}^2 t}{\operatorname{dn} t + k^*}\left[1 + \frac{\beta k^2}{2Y} \frac{\operatorname{cn}^2 t}{\operatorname{dn} t + k^*}\right] \tag{22}$$

in this range and to calculate the zeta-function $\operatorname{zn} t$ by means of its Taylor's series about $t=K$. The latter can be found easily through a power series solution of the ordinary differential equation

$$\frac{d^2}{dt^2} \operatorname{cn}^2 t = 2k^{*2} + (8k^2 - 4) \operatorname{cn}^2 t - 6k^2 \operatorname{cn}^4 t \tag{23}$$

for $\mathrm{cn}^2\, t = k^{-2}(EK^{-1} - k^{*2}) + k^{-2}d(\mathrm{zn}\; t)/dt$. Elsewhere, of course, it is better to represent the Jacobian elliptic functions in terms of theta-functions.

We shall take $N=20$ as the number of explicit solutions p_j of (1) to be used in the present application of the approximation (15) for the stream function ψ. Let

$$(24)\qquad x_1 = \xi - x, \quad x_2 = \xi + x, \quad r_1 = (x_1^2 + y^2)^{1/2}, \quad r_2 = (x_2^2 + y^2)^{1/2}.$$

We choose as our twenty solutions p_j the functions

$$(25)\qquad p_1 = y^2,$$

$$(26)\quad p_j = y^2 \sum_{m=1}^{2} \left[P'_{2j-1}(\nu_m) q'_{2j-1}(\mu_m) + (-1)^j j(2j-1) q'_1(\mu_m) \right], \qquad j = 2, \cdots, 6,$$

$$(27)\quad p_j = 4\xi y^2 \frac{r_1 + r_2}{r_1 r_2[(r_1 + r_2)^2 - 4\xi^2]}, \qquad \xi = .222(j-6),\ j = 7, 8, 9,$$

$$(28)\quad p_j = y^2 \frac{P'_n(x_1/r_1)}{r_1^{n+2}} + y^2 \frac{P'_n(x_2/r_2)}{r_2^{n+2}}, \qquad \xi = .666,\ n = j - 9,\ j = 10, \cdots, 20,$$

where $P'_n(z)$ denotes the derivative of the nth Legendre polynomial and where $q'_n(z)$ is a Legendre function of the second kind defined by

$$(29)\quad \begin{aligned} q'_n(z) = &- \frac{2^{n+2}(n+1)!}{1\cdot 3\cdot 5 \cdots (2n+1)} \left[\frac{1}{z + (z^2+1)^{1/2}} \right]^{n-1} \\ &\cdot \left[\frac{1}{2(z^2+1)^{1/2}\{z + (z^2+1)^{1/2}\}} \right]^{3/2} \\ &\cdot F\left[\frac{3}{2}, -\frac{1}{2}; n + \frac{3}{2}; \frac{1}{2(z^2+1)^{1/2}\{z + (z^2+1)^{1/2}\}} \right] \end{aligned}$$

in terms of the classical hypergeometric series $F[a, b; c; w]$. The function (25) represents uniform flow; the ellipsoidal harmonics (26) represent flows with appropriate square root singularities at the point of separation; the source and sink functions (27) are familiar in the theory of flows past long, slender bodies of revolution; and, finally, the spherical harmonics (28) are designed to smooth out the more rapid oscillations in the error when we interpolate to fit the boundary conditions of our problem.

The functions $p_1, \cdots, p_{20}$ are all symmetric in the y-axis and all vanish on the x-axis. Thus it suffices to impose the boundary condition (11) along the arc $0 \leqq t \leqq K$ of the approximate free boundary curve (16), (17), and it suffices to impose the boundary condition (2) along the segment $x = X$, $0 \leqq y \leqq Y$. Our formulation of the problem will give special attention to the requirement, essential in practice, that the boundary conditions be weighted near the separation point in a manner adapted to the square root singularity of the stream function there. Thus we interpolate at the points $z_l = x_l + iy_l$ on the free boundary corresponding to the values $(l-1)K/36$ of the parameter t, with $l = 1, \cdots, 36$, and we interpolate at the points $z_l = x_l + iy_l$ on the fixed boundary given by the relations

$$x_l = X,\ y_l = [1 - (l-38)^2/24^2]Y, \qquad \text{with } l = 38, \cdots, 62.$$

We let $z_{37} = z_{38}$, since we shall impose the extra boundary condition (14) at that point, and we agree to define the curvature κ to be zero there. At each point z_l we set

$$c_{lj} = \frac{p_j}{y^2}, \qquad d_{lj} = \frac{\partial p_j/\partial n + \kappa p_j}{y(1 + \kappa y)}, \tag{30}$$

and we set

$$b_l = \frac{1}{1 + \kappa y} \text{ for } l = 1, \cdots, 37; \qquad b_l = 0 \text{ for } l = 38, \cdots, 62. \tag{31}$$

We define

$$a_{lj} = d_{lj} \text{ for } l = 1, \cdots, 37; \qquad a_{lj} = c_{lj} \text{ for } l = 38, \cdots, 62. \tag{32}$$

In terms of these quantities, our least squares approximation to the boundary conditions (2), (11), and (14) reduces to the extremal problem

$$\sum_{l=1}^{62} e_l^2 = \text{minimum}, \qquad e_l = \sum_{j=1}^{N} a_{lj}\lambda_j - b_l, \tag{33}$$

which leads in turn to the symmetric system of simultaneous linear equations

$$\sum_{j=1}^{N} \left(\sum_{l=1}^{62} a_{lm} a_{lj} \right) \lambda_j = \sum_{l=1}^{62} a_{lm} b_l, \qquad m = 1, \cdots, N, \tag{34}$$

for the determination of the coefficients λ_j in the representation (15) of the stream function ψ.

We computed the values of the coefficients b_l, c_{lj} and d_{lj} to six or seven significant figures on an IBM Card-Programmed Electronic Calculator, and various tricks of numerical analysis were used to avoid loss of accuracy in the results. The d_{lj} for $39 \leqq l \leqq 61$ and the c_{lj} for $1 \leqq l \leqq 36$ were computed in addition to the a_{lj} because they are needed for the calculation of the drag coefficient (6) and the normal shift (12) correcting the shape of the free boundary. The linear equations (34) were solved for the λ_j on an IBM Type 650 Magnetic Drum Data-Processing Machine, and thus the complete numerical solution of our boundary value problem was obtained.

A few comments should be made about the difficulties encountered in the numerical work. Although the conglomeration of solutions (25), (26), (27), (28) of (1) was selected on the basis of a good deal of experience and practical intuition concerning functions which would best represent an axially symmetric free surface flow, it was nevertheless impossible to avoid the occurrence of a certain amount of cancellation among the numerous terms on the right in our expression (15) for the stream function ψ. Such cancellations are always to be expected when we expand the solution of a boundary value problem defined in a peculiarly shaped domain in terms of functions which bear no direct relationship to that domain. In the present example, these cancellations contributed in the worst cases a loss of as many as two significant figures in our calculations, and this outcome was reflected in relative smallness of the determinant of the system (34). Thus it was necessary to compute the coefficients a_{lj} with exceptional accuracy in the first place in order to obtain meaningful final results.

4. **Interpretation of the data.** For the method sketched in the previous section, it is desirable to check numerically the convergence of the representation (15) as the number N of interpolating functions increases, and it is desirable to check that these functions have been evaluated at a sufficiently large number of points to describe all the oscillations of the error in fitting the boundary conditions. In our calculation we tested both these items by working out the solution of the problem using many different subsets of our basic system (25), (26), (27), (28) of twenty solutions of the differential equation (1). The results gave clear indication of geometric convergence of the procedure and established that the more reliable answers were those corresponding to the smaller values of the least squares error $62^{-1/2}(e_1^2+ \cdots +e_{62}^2)^{1/2}$, as was, of course, to be expected. However, it turned out that the determinant of the system (34) was so small when all twenty interpolating functions were used that it was preferable to omit the nineteenth, which appeared to be the least significant one. In Tables I, II and III we give the complete numerical data compiled from the solution based on the first sixteen interpolating functions, from the solution based on the first eighteen interpolating func-

tions, and from the final solution based on all the interpolating functions except the nineteenth. In this section we shall discuss in detail the conclusions to be drawn from these results and our estimate of the errors involved.

The oscillating character of the distribution of values of the normal derivative $\partial\psi/y\partial n$ computed at the points $z_{39}, \cdots, z_{61}$ along the obstacle appears to be the largest source of error. Such oscillations are to be expected for the type of numerical approximation we have used in calculating the stream function ψ, and they should average out appreciably when we integrate to obtain the drag coefficient C_D from (6). However, since the pressure distribution even fails to be monotonic, we shall carry through an alternate computation of C_D based on the variational formula (7) and compare the answers found by the two different methods in order to establish their validity.

In applying (7), we shall use the value of the cavitation parameter σ calculated on the basis of the stationary boundary condition (11), whereas we use values of the volume V and the virtual mass coefficient a which are obtained from the classical potential flow past the body bounded by the two circular disks $x=\pm X$, $0\leqq y\leqq Y$ and the approximate free streamline (16), (17). The stream function ψ^* of the latter flow is computed from a representation of the form

$$\psi^* = p_1 + \sum_{j=2}^{N} \lambda_j^* p_j \tag{35}$$

with coefficients λ_j^* determined by a least squares approximation to the boundary condition $\psi^*=0$, which is imposed along the entire perimeter of the obstacle. Thus the boundary condition reduces in this case to the system of simultaneous linear equations

$$\sum_{j=2}^{N}\left(\sum_{l=1}^{36} c_{lm}c_{lj} + \sum_{l=38}^{62} c_{lm}c_{lj}\right)\lambda_j^* + \sum_{l=1}^{36} c_{lm} + \sum_{l=38}^{62} c_{lm} = 0 \tag{36}$$

for the λ_j^*. By means of (35) and (36) we transform (7) into the approximate relation

$$\frac{C_D}{1+\sigma} = 11.7737 - \frac{13.3469}{1+\sigma} \tag{37}$$

between the drag coefficient C_D and the cavitation parameter σ. For the actual computation here, we substituted into (35) all the explicit flows (25), (26), (27), (28) except the nineteenth. This gave the result $a=.0435088$, whereas substitution of only the first eighteen flows gave

$a = .0435114$ and substitution of only the first sixteen flows gave $a = .0435182$, which indicates that the numerical coefficients in (37) are exceptionally accurate.

From an earlier calculation [1] based on only ten solutions of (1) and based on a poorer choice of the stretching parameter β appearing in our approximation (16), (17) of the free boundary, we obtain the alternate numerical form

$$\frac{C_D}{1+\sigma} = 11.1910 - \frac{12.6498}{1+\sigma} \tag{38}$$

of the variational formula (7), which is useful as a check on the reliability of our results because it is derived from data independent of the computations discussed in this article.

Theoretically, both formula (37) and formula (38) should yield lower bounds on the drag coefficient C_D when σ is given, but the errors in the data presented in the tables overshadow this effect. It is consistent with the variational theory that in each table the value of $C_D/(1+\sigma)$ derived from (37) exceeds the corresponding value derived from (38), since (38) involves a less accurate choice of the free boundary. However, even in Table III the value of $C_D/(1+\sigma)$ found from (6) by direct application of Simpson's rule lies between the two values based on the variational formula (7), and the magnitude of the discrepancy here must serve as an estimate of the error occurring in our numerical evaluation of the cavitation parameter σ and of the integral in (6). Within the margin of accuracy necessary for a physically significant description of the flow, the general extent of agreement between the results based on (6) and the results based on (7) suffices, on the other hand, to establish the reliability of the data.

A study of Tables I, II and III shows that there is a quite definite trend in the numerical data as the number N of interpolating functions is increased. Indeed, as N increases we observe that the corresponding values of the cavitation parameter $\sigma = 1/4\lambda_1^2 - 1$ and of the scaled drag coefficient $C_D/(1+\sigma)$ decrease, as does also the cavity radius d, which is defined to be the value of y at the point on the free streamline where $x=0$. A crude extrapolation based on the three examples presented in the tables indicates that the errors in the values of σ and d/Y listed in Table III do not exceed 1 per cent and that the true value of $C_D/(1+\sigma)$ must lie between .85 and .88. It is fair to say that the best estimates of these quantities which can be deduced from all the information compiled in our calculation are given by

$$\sigma = .2235, \qquad \frac{d}{Y} = 2.30, \qquad \frac{C_D}{1+\sigma} = .865, \tag{39}$$

and it is quite probable that the errors incurred here are less than 1/2 per cent.

We have presented in the tables values of x, y and $-\psi/y^2$ at the points $z_1, \cdots, z_{36}$ along the curve (16), (17), and it is an easy matter to compute from this information the normal shift (12) which transforms the first guess (16), (17) of the shape of the free streamline into our final approximation to that curve. The quantity $-\psi/y^2$ can, indeed, be interpreted as a scaled normal shift. We note that it is everywhere positive, indicating that the true free boundary lies above our first guess, while at the same time the normal shift listed in Table III remains small enough so that the final curve lies well below the corresponding plane free streamline, in agreement with general theory [1]. The fact that when the index l increases the values of $-\psi/y^2$ increase significantly before they finally fall off toward zero near the separation point has the interesting interpretation that the axially symmetric free streamline is flatter at the wide section of the cavity than an affine map of the corresponding plane free streamline. This conclusion is consistent with Levinson's asymptotic formula [4] describing the shape of an infinite axially symmetric cavity. We remark that a quite accurate numerical solution of the mixed boundary value problem stated in §2 was required in order to provide a significant estimate of the normal shift.

Our calculation (39) of the scaled drag coefficient $C_D/(1+\sigma)$ combines with earlier investigations [1] to furnish a fairly complete picture of the dependence of this important physical quantity on the cavitation parameter σ. It was shown previously that $C_D/(1+\sigma)$ increases monotonically from the approximate value .827 to the exact value 1 as σ increases from zero to infinity, and that the derivative of $C_D/(1+\sigma)$ with respect to σ vanishes at $\sigma=0$. The result (39) now serves to describe more precisely the rate of growth of the drag coefficient for intermediate values of σ, and, indeed, it would be possible to base on the above data an empirical formula for $C_D/(1+\sigma)$ in terms of σ which would be quite accurate in the physically relevant range.

It is interesting at this stage to compare the plot of the drag coefficient obtained in the above manner with the corresponding graphs of experimental measurements. Such a comparison shows that our figures exceed the experimental values by roughly 2 per cent, an outcome which actually signifies substantial agreement of theory with experiment. Furthermore, the noticeable increase in $C_D/(1+\sigma)$ from .827 to .865 in the interval between $\sigma=0$ and $\sigma=.2235$, which is considerably bigger than the corresponding increment encountered in the two-dimensional flow theory, explains a relatively large slope in the experimentally observed values of C_D which has appeared heretofore to be rather puzzling.

The slight excess in the calculated values of $C_D/(1+\sigma)$ over the measured values may possibly be due to the effect of gravity, since experiments executed at high pressure indicate that an analogous discrepancy with theory (cf. [1]) in the determination of the contraction coefficient of a jet issuing from a circular orifice is caused by gravity. In fact, both the drag coefficient and the contraction coefficient can be expressed linearly in terms of an integral of the form

$$J = \int \left(\frac{\partial\psi}{\partial n}\right)^2 \frac{dy}{y} \tag{40}$$

involving the stream function ψ of the relevant flow, and while a minus sign appears before the integral in the case of the drag coefficient, the corresponding sign for the contraction coefficient is plus. Thus, since the accepted experimental value .61 of the contraction coefficient exceeds by a few per cent the calculated value .58, the apparent effect of introducing gravity is to increase the value assigned to J, and if J were correspondingly augmented in a measurement of the drag coefficient influenced by gravity, we would expect the result to be a value slightly smaller than that given by (39), an outcome consistent with the actual facts.

In closing, we should point out that the drag coefficient C_D and the cavitation parameter σ can be estimated from the variational formula (7) independently of the stationary boundary condition (11) formulated in §2. This is done simply by solving the two simultaneous equations (37) and (38), based on different approximations to the shape of the free surface, for σ and C_D. The answers obtained in this manner are $\sigma = .20$ and $C_D/(1+\sigma) = .62$, and the large errors involved are merely a consequence of the sizeable cancellations which prove to be unavoidable in such an approach. The only advantage which can be claimed for this method is that it depends only on calculation of the virtual mass coefficient a, which is probably the easiest and most accurate numerical step we have undertaken.

References

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TABLE I

l	x_l	y_l	e_l	$-\psi/y^2$	λ_l	l	e_l	$\partial\psi/y\partial n$
1	.00000	.33099	−.00304	.0416	.451052	38	−.00187	
2	.07168	.33033	−.00140	.0421	$-.732372\ 10^{-1}$	39	−.00113	.7901
3	.14257	.32837	.00255	.0433	$-.127971\ 10^{-1}$	40	.00087	.6204
4	.21190	.32516	.00555	.0441	$-.242056\ 10^{-2}$	41	.00350	.4934
5	.27898	.32079	.00306	.0437	$-.136522\ 10^{-3}$	42	.00596	.4159
6	.34319	.31536	−.00448	.0425	$-.154673\ 10^{-3}$	43	.00751	.3852
7	.40404	.30902	−.00594	.0428	$.399920\ 10^{-2}$	44	.00768	.3897
8	.46114	.30189	.00534	.0448	−.137642	45	.00646	.4126
9	.51423	.29414	.00707	.0455	.103587	46	.00426	.4365
10	.56315	.28591	−.00913	.0442	$-.309547\ 10^{-1}$	47	.00179	.4473
11	.60787	.27735	−.00081	.0455	$-.239546\ 10^{-2}$	48	−.00022	.4379
12	.64841	.26860	.01219	.0472	$-.437344\ 10^{-3}$	49	−.00123	.4085
13	.68491	.25977	−.00817	.0457	$-.416463\ 10^{-4}$	50	−.00112	.3657
14	.71753	.25098	−.00848	.0460	$-.832451\ 10^{-5}$	51	−.00016	.3186
15	.74650	.24231	.01274	.0483	$-.549745\ 10^{-6}$	52	.00107	.2757
16	.77206	.23385	.01091	.0481	$-.961096\ 10^{-7}$	53	.00194	.2417
17	.79447	.22567	−.00775	.0459	.000000	54	.00203	.2164
18	.81401	.21781	−.01673	.0445	.000000	55	.00132	.1960
19	.83094	.21031	−.01005	.0450	.000000	56	.00016	.1754
20	.84552	.20321	.00337	.0466	.000000	57	−.00089	.1509
21	.85799	.19652	.01448	.0477		58	−.00135	.1218
22	.86858	.19025	.01941	.0477		59	−.00105	.0901
23	.87752	.18442	.01831	.0462		60	−.00021	.0584
24	.88500	.17903	.01314	.0434		61	.00066	.0286
25	.89119	.17407	.00606	.0396		62	.00104	
26	.89626	.16955	−.00117	.0353				
27	.90036	.16545	−.00742	.0307				
28	.90363	.16177	−.01210	.0261				
29	.90617	.15850	−.01501	.0217				
30	.90811	.15565	−.01622	.0176				
31	.90953	.15318	−.01597	.0139				
32	.91054	.15111	−.01455	.0107				
33	.91120	.14943	−.01225	.0078				
34	.91159	.14813	−.00938	.0055				
35	.91179	.14720	−.00627	.0036				
36	.91187	.14664	−.00342	.0024				
37	.91188	.14646	−.01283					

$\sigma = .22882,\quad d/Y = 2.354$

$$\frac{C_D}{1+\sigma} = \begin{cases} .8881 & \text{by (6)} \\ .9121 & \text{by (37)} \\ .8967 & \text{by (38)} \end{cases}$$

$$62^{-1/2}(e_1^2 + \cdots + e_{62}^2)^{1/2} = .00832$$

TABLE II

l	x_l	y_l	e_l	$-\psi/y^2$	λ_l
1	.00000	.33099	−.00159	.0271	.451680
2	.07168	.33033	−.00061	.0274	−.107688
3	.14257	.32837	.00158	.0280	$-.235624\ 10^{-1}$
4	.21190	.32516	.00268	.0284	$-.505784\ 10^{-2}$
5	.27898	.32079	.00032	.0281	$-.867247\ 10^{-3}$
6	.34319	.31536	−.00320	.0278	$-.271143\ 10^{-3}$
7	.40404	.30902	−.00054	.0286	$.221253\ 10^{-2}$
8	.46114	.30189	.00431	.0296	−.203479
9	.51423	.29414	−.00252	.0293	.171748
10	.56315	.28591	−.00153	.0298	$-.462914\ 10^{-1}$
11	.60787	.27735	.00506	.0307	$-.408437\ 10^{-2}$
12	.64841	.26860	−.00582	.0302	$-.712788\ 10^{-3}$
13	.68491	.25977	.00349	.0312	$-.836584\ 10^{-4}$
14	.71753	.25098	.00267	.0314	$-.169229\ 10^{-4}$
15	.74650	.24231	−.00723	.0307	$-.176830\ 10^{-5}$
16	.77206	.23385	.00194	.0315	$-.341559\ 10^{-6}$
17	.79447	.22567	.00798	.0320	$-.230466\ 10^{-7}$
18	.81401	.21781	.00048	.0310	$-.366784\ 10^{-8}$
19	.83094	.21031	−.00760	.0296	.000000
20	.84552	.20321	−.00777	.0291	.000000
21	.85799	.19652	−.00190	.0292	
22	.86858	.19025	.00488	.0294	
23	.87752	.18442	.00916	.0291	
24	.88500	.17903	.01000	.0280	
25	.89119	.17407	.00803	.0262	
26	.89626	.16955	.00443	.0238	
27	.90036	.16545	.00036	.0211	
28	.90363	.16177	−.00333	.0181	
29	.90617	.15850	−.00615	.0151	
30	.90811	.15565	−.00790	.0123	
31	.90953	.15318	−.00858	.0097	
32	.91054	.15111	−.00832	.0074	
33	.91120	.14943	−.00731	.0054	
34	.91159	.14813	−.00576	.0037	
35	.91179	.14720	−.00390	.0024	
36	.91187	.14664	−.00212	.0015	
37	.91188	.14646	−.00442		

l	e_l	$\partial\psi/y\partial n$
38	−.00115	
39	−.00062	.8203
40	.00078	.6723
41	.00251	.5649
42	.00393	.5024
43	.00451	.4791
44	.00403	.4812
45	.00271	.4912
46	.00107	.4936
47	−.00027	.4786
48	−.00083	.4450
49	−.00051	.3990
50	.00038	.3504
51	.00123	.3085
52	.00152	.2776
53	.00102	.2557
54	.00001	.2363
55	−.00089	.2128
56	−.00105	.1824
57	−.00037	.1479
58	.00066	.1147
59	.00119	.0862
60	.00072	.0605
61	−.00034	.0326
62	−.00091	

$\sigma = .22540, \quad d/Y = 2.321$

$$\frac{C_D}{1+\sigma} = \begin{cases} .8697 & \text{by (6)} \\ .8818 & \text{by (37)} \\ .8680 & \text{by (38)} \end{cases}$$

$$62^{-1/2}(e_1^2 + \cdots + e_{62}^2)^{1/2} = .00426$$

TABLE III

l	x_l	y_l	e_l	$-\psi/y^2$	λ_l	l	e_l	$\partial\psi/y\partial n$
1	.00000	.33099	.00035	.0230	.451839	38	−.00010	
2	.07168	.33033	.00022	.0230	−.138782	39	.00044	.8035
3	.14257	.32837	−.00020	.0230	−.346617 10^{-1}	40	.00180	.6451
4	.21190	.32516	−.00058	.0231	−.822686 10^{-2}	41	.00330	.5411
5	.27898	.32079	−.00013	.0235	−.179683 10^{-2}	42	.00414	.4957
6	.34319	.31536	.00086	.0240	−.509280 10^{-3}	43	.00379	.4974
7	.40404	.30902	.00032	.0242	−.317849 10^{-2}	44	.00223	.5237
8	.46114	.30189	−.00112	.0245	.114032 10^{-1}	45	.00002	.5480
9	.51423	.29414	.00081	.0251	−.380732 10^{-1}	46	−.00191	.5490
10	.56315	.28591	.00023	.0254	−.743505 10^{-3}	47	−.00268	.5173
11	.60787	.27735	−.00104	.0257	.847660 10^{-3}	48	−.00188	.4579
12	.64841	.26860	.00189	.0262	−.433964 10^{-4}	49	.00016	.3868
13	.68491	.25977	−.00237	.0262	.205273 10^{-4}	50	.00243	.3235
14	.71753	.25098	.00226	.0268	−.227767 10^{-5}	51	.00367	.2824
15	.74650	.24231	−.00013	.0267	.489988 10^{-6}	52	.00303	.2657
16	.77206	.23385	−.00312	.0265	−.104352 10^{-6}	53	.00066	.2628
17	.79447	.22567	.00251	.0268	.686082 10^{-8}	54	−.00224	.2568
18	.81401	.21781	.00374	.0266	−.352631 10^{-8}	55	−.00392	.2338
19	.83094	.21031	−.00133	.0257	.000000	56	−.00309	.1920
20	.84552	.20321	−.00504	.0248	−.601794 10^{-10}	57	.00001	.1427
21	.85799	.19652	−.00389	.0242		58	.00340	.1021
22	.86858	.19025	.00024	.0240		59	.00454	.0778
23	.87752	.18442	.00428	.0236		60	.00227	.0619
24	.88500	.17903	.00639	.0228		61	−.00172	.0380
25	.89119	.17407	.00621	.0214		62	−.00378	
26	.89626	.16955	.00431	.0196				
27	.90036	.16545	.00154	.0174				
28	.90363	.16177	−.00130	.0150				
29	.90617	.15850	−.00368	.0125				
30	.90811	.15565	−.00531	.0101				
31	.90953	.15318	−.00610	.0079				
32	.91054	.15111	−.00610	.0058				
33	.91120	.14943	−.00542	.0041				
34	.91159	.14813	−.00420	.0025				
35	.91179	.14720	−.00262	.0013				
36	.91187	.14664	−.00101	.0005				
37	.91188	.14646	−.00201					

$\sigma = .22454, \quad d/Y = 2.312$

$$\frac{C_D}{1+\sigma} = \begin{cases} .8624 & \text{by (6)} \\ .8742 & \text{by (37)} \\ .8608 & \text{by (38)} \end{cases}$$

$$62^{-1/2}(e_1^2 + \cdots + e_{62}^2)^{1/2} = .00302$$