RESEARCH PROBLEMS


Let \( f(x) \) be a monotone increasing function of \( x \) with positive continuous derivative for \( x \geq 0 \), with \( f(0) = 0, f(\infty) = \infty \). Consider the equation

\[
f(x) = y,
\]
possessing the unique solution \( x = f^{-1}(y) \) for \( y \geq 0 \). Let

\[
x_{n+1} = x_n + \frac{y - f(x_n)}{f'(x_n)}, \quad x_0 = z,
\]
be the sequence of successive approximations to \( f^{-1}(y) \) furnished by Newton’s method.

Determine \( z = z(a, b, n) \) so that

\[
\max_{x_n \in I} |x_n - f^{-1}(y)|
\]
is a minimum, where \( 0 < a < b < \infty \), and determine the asymptotic behavior of \( z(a, b, n) \) as \( n \to \infty \).

For \( f(x) = x^4 \), it is known that \( z(a, b, n) \to (ab)^{1/4} \) as \( n \to \infty \). (Received November 26, 1956.)


At the present time, there is no systematic technique for solving the problem of maximizing the linear form \( L(x) = \sum_{i=1}^{N} a_i x_i \) subject to the constraints \( \sum_{j=1}^{M} b_{ij} x_j \leq c_i \), \( i = 1, 2, \ldots, M \), where the \( a_i \) and \( b_{ij} \) are positive integers, or zero, and the \( x_j \) are constrained to be positive integers or zero. On the other hand, if this constraint on integral solutions is removed, the solution is readily obtained for small \( M \), and there exist effective algorithms for large \( M \).

For the case \( M = 1 \), let \( f_N(c) \) denote the maximum of \( L(x) \) under integral constraints and \( z_N(c) \) denote the solution under the constraint \( x_i \geq 0 \). Define the function

\[
\phi(N) = \sup_{a_i, b_{ij} \geq 1} \left[ \sup_{c_i, b_{ij}} \frac{f_N(c)}{z_N(c)} \right].
\]

What is the order of magnitude of \( \phi(N) \) as \( N \to \infty \), and in particular, is it bounded?

Consider the corresponding problem for general \( M \) where

\[
\phi_M(N) = \sup_{a_i, b_{ij} \geq 1} \left[ \sup_{c_i, b_{ij}, c_M} \frac{f_N(c_1, c_2, \ldots, c_M)}{z_N(c_1, c_2, \ldots, c_M)} \right]
\]
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