Geometric integration Theory by Hassler Whitney. Princeton, Princeton University Press, 1957. 15+387 pp. $8.50.

This book deals mainly with the following problem connecting algebraic topology with analysis and differential geometry: Characterize those cochains which, together with their coboundaries, can be obtained by the integration of differential forms.

Here is one answer (Chapter IX, Theorem 5A) to this question, which impresses me as the most interesting theorem in the book. (This theorem was first proved in the 1948 Harvard Ph.D. thesis of the author's student J. H. Wolfe.) Consider all real valued r dimensional cochains X which are defined on the finite rectilinear simplicial chains in an open subset R of Euclidean n-space, and for which there exists a real number M such that if σ is any r dimensional simplex in R, then $|X(\sigma)|$ does not exceed M times the r dimensional measure of the point set spanned by σ; define the norm $|X|$ as the least such number M. If both X and its $r+1$ dimensional coboundary dX have finite norm, then X and dX can be computed by Lebesgue integration of bounded and measurable r and $r+1$ dimensional differential forms, defined almost everywhere in R, and almost everywhere in each r and $r+1$ dimensional simplex in R respectively.

This theorem is a generalization of Rademacher's classical result asserting the differentiability almost everywhere of a function satisfying a Lipschitz condition. In fact, if $r=0$ and R is convex, then X is a real valued function, $|X|$ is the supremum of X, and $|dX|$ is the best Lipschitz constant for X. The proof of the present general theorem adds significant new features to the classical argument. From the finiteness of $|X|$ and $|dX|$ it follows that X is alternating, and that X is additive with respect to subdivision. Hence the classical theory of finitely additive set functions implies that in each r dimensional plane the cochain X is the indefinite integral of its bounded measurable derivative with respect to r dimensional measure. The main difficulty is to show that this derivative depends continuously and even linearly, in the sense of Grassmann algebra, on the directions of all the r dimensional planes through a point. The proof of continuity uses the finiteness of $|X|$ and $|dX|$ to show that X has nearby values on the bottom simplex and the (not necessarily parallel) top simplex of a simplicial prism of small height. The proof of linearity uses the finiteness of $|dX|$ to verify the conditions of a known algebraic criterion for the multilinearity of a homogeneous alternating function.

Attaching to each cochain X of the above type the new "flat" norm

$$|X|^\flat = \sup \{ |X|, |dX| \}$$

one obtains a Banach space in which the coboundary operator is continuous. The resulting cohomology spaces are isomorphic with the usual cohomology groups of the open set R with real coefficients. This isomorphism is established explicitly, following the classical approach of de Rham, and without use of the general theory of Leray and H. Cartan. Cochains of finite flat norm are studied also, with similar results, for the case in which R is any Euclidean polyhedron.

The space of cochains of finite flat norm is shown to possess a unique suitable product, bounded with respect to the flat norm. This product corresponds to Grassmann multiplication of representative differential forms, and is distinct from the ordinary simplicial cup product. Of course the resulting cohomology ring is isomorphic with the usual ring, as a consequence of well known uniqueness theorems.

Since cochains are linear operators on chains, it is natural to seek to represent the Banach space of cochains of finite flat norm as the conjugate space of a suitably normed vector space of chains. The author's explicit solution of this problem may be thought of in terms of the following general situation:

Let V be a normed real vector space with conjugate space V^*, and norm $V\times V$ and $V^*\times V^*$ by

$$|(x, y)| = |x| + |y| \quad \text{for } x, y \in V,$$
$$|(\xi, \eta)| = \sup\{|\xi|, |\eta|\} \quad \text{for } \xi, \eta \in V^*,$$

so that $V^*\times V^*$ acts as conjugate space of $V\times V$ through the pairing

$$(\xi, \eta)\cdot(x, y) = \xi(x) + \eta(y) \text{ for } x, y \in V \text{ and } \xi, \eta \in V^*.$$

Suppose T: $V\rightarrow V$ is a closed (but not necessarily continuous) linear transformation with the conjugate T^*. [Observe that

$$T = \{(x, y) \mid T(x) - y = 0\}$$

is a closed subset of $V\times V$ if and only if T is the annihilator of its own annihilator

$$U = \{(\xi, \eta) \mid \xi + T^*(\eta) = 0\}$$

in $V^*\times V^*$, and also if and only if the domain of T^* has a trivial annihilator in V.] The linear transformation

$$f\colon V\times V\rightarrow V, f(x, y) = T(x) - y \text{ for } x, y \in V,$$

whose kernel is T, induces a new norm on V defined by

$$|a|' = \inf_{f(x,y)=a} |(x, y)| = \inf_{x\in V} (|x| + |T(x) - a|) \text{ for } a \in V.$$

Then f^* maps the new conjugate space of V isometrically onto U, which is in turn isometric to the domain of T^* with the new norm

$$|\eta|' = \sup \{ |\eta|, |T^*(\eta)| \} \text{ for } \eta \in \text{domain } T^*.$$

In this way the domain of T^* becomes the conjugate space of V, with new norms but with the old pairing; furthermore T becomes continuous.

In the present instance let V be the vector space obtained from the alternating finite simplicial chains in the open set R, with real coefficients, by identifying chains with their subdivisions and by neglecting degenerate chains. Every element of V, called a "polyhedral chain," may be represented as a finite linear combination of nonoverlapping nondegenerate simplexes with real coefficients. If x is an r dimensional polyhedral chain so represented, then $|x|$ equals the sum of the r dimensional measures of the point sets spanned by the simplexes, each multiplied by the absolute value of its coefficient. Then V^* consists of all alternating cochains X which are additive with respect to subdivision and for which $|X|$ is finite. Let T be the boundary operator ∂ of V. Then T^* is the restriction of the coboundary operator d to the set of those cochains X for which $|X|^\flat$ is finite. Clearly the domain of T^* has a trivial annihilator in V. It follows that the space of cochains of finite flat norm is the conjugate space of the space V of polyhedral chains with respect to the "flat" norm defined by

$$|a|^\flat = \inf_{x \in V} (|x| + |\partial x - a|) \text{ for } a \in V.$$

The completion of the space of polyhedral chains with respect to its flat norm has not been characterized by a representation theorem. However, it is shown that to each Lipschitzian singular simplex corresponds an element of this completion, defined as the limit of reasonably inscribed polyhedral chains. It follows that Lipschitzian maps induce suitable homomorphisms of the spaces of cochains, and of the completed spaces of chains, with finite flat norm.

Besides the "flat" theory described here, the author considers in detail a "sharp" theory dealing with cochains representable by Lipschitzian differential forms. Among other related topics treated are measure theoretic representation theorems for certain types of chains.

The first four chapters contain a very useful collection of classical material. Here the author gives an original geometric treatment of the basic properties of differential forms and their integration, of his own theorem on the embedding of differentiable manifolds in Euclidean space, of the triangulation of differentiable manifolds, and of the theorem of de Rham.

Throughout the book one finds numerous simple examples which are most helpful in explaining the author's motivation. One may hope that his enthusiasm will continue and will inspire further contributions in this field.

HERBERT FEDERER

Handbuch der Physik, Band 1, *Mathematische Methoden* 1. Ed. by S. Flügge. Springer, Berlin, 1956. vii+364 pp. DM72.

This is the first of the two volumes of the encyclopedia of physics devoted to mathematical methods. The book starts with a 90 page section on analysis. This presents the principal definitions and theorems relating to calculus, ordinary differential equations, and the analysis of real and complex numbers. There is much exposition of prerequisite notions of algebra and trigonometry, as well as an appendix on the Lebesgue integral. There follow two sections of about 30 pages each on partial differential equations and elliptic functions. Although all these were contributed by the same author, J. Lense, the two specialized sections contain an outline of the theory as well as a collection of statements. For example, the discussion of elliptic functions starts out from Weierstrass's point of view, and later leads into the results of Legendre and Jacobi.

The section on special functions of some 70 pages was written by J. Meixner. This deals principally with functions related to the hypergeometric function and its limiting cases, and those related to Mathieu functions. This section includes many indications of proofs, and the classification proceeds by general methods, including some of the ideas of Truesdell, rather than by individual functions. But cross references are given to facilitate the study of any one class of function.

The final section of some 140 pages was written by F. Schlögl. This begins by treating orthogonal functions, integral equations, and the calculus of variations. It then proceeds to the discussion of boundary value problems of partial differential equations as such.

In this entire volume there are some, but relatively few, footnote references. However each author has included a short bibliography of basic texts and a few articles at the end of his contribution. These three lists have some items in common.