ON THE PARALLELIZABILITY OF THE SPHERES

BY R. BOTT AND J. MILNOR

Communicated by H. Samelson, February 13, 1958

(The following note consists of excerpts from two letters.)
(Milnor to Bott; December 23, 1957.)

. . . Hirzebruch tells me that you have a proof of his conjecture that the Pontrjagin class $p_k$ of a $GL_m$-bundle over the sphere $S^{4k}$ is always divisible by $(2k - 1)!$. I wonder if you have noted the connection of this result with classical problems, such as the existence of division algebras, and the parallelizability of spheres.

According to Wu the Pontrjagin classes of any $GL_m$-bundle, reduced modulo 4, are determined by the Stiefel-Whitney classes of the bundle. (See On the Pontrjagin classes III, Acta Math. Sinica vol. 4 (1954) in Chinese.) The proof makes use of the Pontrjagin squaring operation, together with the coefficient homomorphism $i: \mathbb{Z}_2 \to \mathbb{Z}_4$. Although I do not know the exact formula which Wu obtains, the following special case is not hard to prove:

**Lemma.** If the Stiefel-Whitney classes $w_1, w_2, \cdots, w_{4k-1}$ of a $GL_m$-bundle are zero then the Pontrjagin class $p_k$, reduced modulo 4, is equal to $i^*w_{4k}$.

For a bundle over $S^{4k}$ this means that $w_{4k}$ is zero if and only if $p_k$ is divisible by 4. Now if you can prove that $p_k$ is divisible by $(2k - 1)!$ it will follow that $w_{4k}$ must be zero, whenever $k \geq 3$.

**Theorem.** There exists a $GL_m$-bundle over $S^n$ with $w_n \neq 0$ only if $n$ equals 1, 2, 4 or 8.

**Proof.** Wu has shown that such a bundle can only exist if $n$ is a power of 2. But the above remarks show that the cases $n = 16, 32, \cdots$ cannot occur.

**Corollary 1.** The vector space $\mathbb{R}^n$ possesses a bilinear product operation without zero divisors only for $n$ equal to 1, 2, 4 or 8.

**Proof.** Given such a product operation the map $S^{n-1} \to GL_n$ defined by $x \to (\text{left multiplication by } x)$ gives rise to a $GL_n$-bundle over $S^n$ for which it can be shown that $w_n \neq 0$.

**Corollary 2.** The sphere $S^{n-1}$ is parallelizable only for $n-1$ equal to 1, 3 or 7.

**Proof.** Given linearly independent vector fields $v_1(x), \cdots, v_{n-1}(x)$,
on $S^{n-1}$, the correspondence

$$x \rightarrow (x, v_1(x), \cdots, v_{n-1}(x))$$

carries $S^{n-1}$ into the Stiefel manifold of $n$-frames in $R^n$. Identifying this space with $GL_n$, we again obtain a $GL_n$-bundle over $S^n$ with $w_n \neq 0 \cdots$

(Bott to Milnor, January 6, 1958.)

... Here is what I can show:

**Theorem.** Let $B = BU$ be the universal base-space of the infinite unitary group. Then the image of $\pi_{2n}(B)$ in $H_{2n}(B)$ is divisible by precisely $(n-1)!$.

This then refines the result of Borel-Hirzebruch that these classes are divisible by $(n-1)!$ except for the prime 2, [3], and confirms their conjecture. Because the Pontryagin classes are in the last analysis pre-images of classes in $BU$, it follows that for any $GL_n(R)$ bundle over $S^k$, $p_k$ is divisible by $(2k-1)!$. This is all you needed.

The precise divisibility of $p_k$, for a real bundle over $S^k$, is actually given by:

$$p_k = 0, \mod (2k-1)!, \quad k \text{ even},$$

$$p_k \equiv 0 \mod (2k-1)!, \quad k \text{ odd}.$$ 

This is seen by considering the fibering $U/O \rightarrow B_O \rightarrow BU$.

The theorem follows from the fact, that if $\Omega = \Omega SU$ is the loopspace on $SU$, then there exists a homotopy equivalence $f: B \rightarrow \Omega$, as was announced in [1] and is proved in [2]. By standard theory the double suspension, $S$, from $\Omega$ into $B$, defines a homomorphism $\pi_{2k}(\Omega) \rightarrow \pi_{2k+2}(B)$ which is bijective for dimensions $\geq 1$.

Let $\lambda = f \circ OS$. It is then clear that:

$$\pi_{2k}(\Omega) = \lambda^{k-1}(\pi_2(\Omega)).$$

Now in [2] the Hopf algebra $H_*(\Omega)$ is described. It turns out that $H_*(\Omega) = \mathbb{Z}[\sigma_1, \sigma_2, \cdots]$, dim $\sigma_i = 2i$, the diagonal map being: $\Delta_* \sigma_i = \sum \sigma_s \otimes \sigma_t$; $s+t=i$; $\sigma_0 = 1$. Hence the primitive subspace, $P_*$, is generated by elements $\{p_n\}, \quad n = 1, 2, \cdots$, which are inductively determined by the relation:

$$p_n - p_{n-1} \cdot \sigma_1 + p_{n-2} \cdot \sigma_2 - \cdots + \pm n \sigma_n = 0, \quad n = 1, 2, \cdots.$$ 

Let $\lambda_*$ be the homomorphism corresponding to $\lambda$ in homology. It will preserve spherical classes, and annihilate decomposable elements. It therefore follows from (2) that $\lambda_* p_i = \pm i \lambda_* \sigma_i$. As the spherical classes generate $P_*$ (over the rationals, $SU$ is a product of odd spheres!) this
relation implies that $\lambda_{k-1}^{k-1}(H^2(\Omega))$ is divisible by at least $(k-1)!$. By (1) it follows that the spherical classes in dimension $2k$ are divisible by at least $(k-1)!$. This is the best bound because it is not hard to see that $\lambda$ is not divisible on all of $H^*(\Omega)$.

An easy corollary of the theorem is that $\pi_{2n}(U_n) = \mathbb{Z}/n!\mathbb{Z}$. Kervaire also has decided the parallelizability question. He uses this formula as his starting point . . . .

REFERENCES

2. ———, The Pontryagin ring of $\Omega G$ (to be published in Michigan Math. J.).
3. A. Borel and F. Hirzebruch: Characteristic classes and homogeneous spaces (to be published in Amer. J. Math.).