AN UNSHELLABLE TRIANGULATION OF A TETRAHEDRON

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A triangulation $K$ of a tetrahedron $T$ is shellable if the tetrahedra $K_1, \cdots, K_n$ of $K$ can be so ordered that $K_i \cup K_{i+1} \cup \cdots \cup K_n$ is homeomorphic to $T$ for $i = 1, \cdots, n$. Sanderson [Proc. Amer. Math. Soc. vol. 8 (1957) p. 917] has shown that, if $K$ is a Euclidean triangulation of a tetrahedron then there is a subdivision $K'$ of $K$ which is shellable; and he raises the question of the existence of a Euclidean triangulation of a tetrahedron which is unshellable. Such a triangulation will be described here.

Let $T$ be a tetrahedron each of whose edges has length 1.

We will describe a nontrivial Euclidean triangulation $K$ of $T$ such that, if $R$ is any tetrahedron of $K$, then the closure of $(T - R)$ is not homeomorphic to $T$.

I. Construction of $K$: Let $X_1, X_2, X_3, \text{ and } X_4$ be the vertices of $T$.

The possible values for the letters $i$ and $j$ are 1, 2, 3, and 4 and addition involving $i$ or $j$ will be modulo 4.

For each $i$, let $F_i$ denote the face of $T$ opposite $X_i$, and let $U_i$ be the midpoint of the interval $X_iX_{i+2}$. Observe that $U_1 = U_3$ and $U_2 = U_4$.

Let $\epsilon$ be the length of the shortest side of a triangle whose longest side is of length 1 and two of whose angles are $1^\circ$ and $60^\circ$.

For each $i$, let $Y_i$ denote the point of $F_{i+1}$ at a distance $(3^{1/2}/2)\epsilon$ from $X_i$ such that the angle $Y_iX_iX_{i+2}$ is $1^\circ$.

For each $i$, let $Z_i$ denote the point of $F_{i+2}$ such that the angle $Z_iX_{i+1}X_i$ is $1^\circ$ and the angle $Z_iX_{i+1}X_i$ is $1^\circ$.

The fourteen vertices of our triangulation $K$ are the points $X_i, Y_i, Z_i, \text{ and } U_i$. It can be shown that no triangulation which has less than 14 vertices has the desired property.

The tetrahedra of our triangulation $K$ are the tetrahedra of the forms:

1. $X_iZ_iX_{i+1}Y_i$,
2. $X_iZ_{i+1}X_{i+1}Y_i$,
3. $Z_iZ_{i+1}X_{i+1}Y_i$,
4. $Z_iZ_{i+1}X_{i+1}Y_{i+1}$,
5. $Z_iZ_{i+1}Z_{i+2}$,
6. $Z_iZ_{i+1}Y_iZ_{i+2}$,
7. $X_iZ_{i+1}Y_iZ_{i+2}$,
8. $X_iZ_{i+1}Y_{i+2}Z_{i+3}$,
9. $X_iU_iY_{i+2}Z_{i+3}$,
II. Checking the construction: The best method of doing this is to draw a big picture and label the vertices.

It is easy to check that for each tetrahedron $R$ of $K$, the closure of $(T - R)$ is not homeomorphic to $T$.

In order to check that $K$ is a triangulation, first observe that, for each $i$, the tetrahedra (1), (2), (3), and (4) fit together and form a thin rod having the triangles $X_iY_iZ_i$ and $X_{i+1}Y_{i+1}Z_{i+1}$ for its ends; the union of these rods forms a torus running along the edges $X_iX_{i+1}$.

When (5) is added to this torus the remainder of $T$ is divided into two congruent pieces each containing pieces of $T$ along the faces $F_i$ and $F_{i+2}$. After (6), (7), and (8) are added to the first five types there is only a small strip around $X_iX_{i+2}$ remaining of $T$; (9) and (10) complete the faces of $T$ and (11) fills in the final space.

To see that the tetrahedra all nest together properly in the order just described, the following facts will be useful. Fact A is needed for the “rods.” Fact B is needed for (3). Facts C and D are needed as assurance that none of the tetrahedra of types (5) through (11) intersect the interior of the torus. Fact E is needed to show that (7) does not intersect either (2) or (6). And facts F, G, and H are needed to show that the tetrahedra of types (6) through (11) for $i = 1$ do not intersect the tetrahedra of the same types for $i = 3$. The facts can be easily proved using the definitions of $\epsilon$, $Y_i$, and $Z_i$.

(A) The plane $X_iY_iZ_i$ separates $X_{i+1}$, $Y_{i+1}$, $Z_{i+1}$ from $X_{i-1}$, $Y_{i-1}$, and $Z_{i-1}$.

(B) The points $X_i$ and $Y_i$ are on the same side of the plane $X_{i+2}Z_{i+2}$.

(C) The plane $Y_iZ_iZ_{i+3}$ separates $X_i$ and $X_{i+3}$ from $U_i$, $X_{i+2}$, $Y_{i+2}$, $Z_{i+2}$, $X_{i+1}$, $Y_{i+1}$, and $Z_{i+1}$.

(D) The plane $Y_iZ_iZ_{i+1}$ separates $X_i$ and $X_{i+1}$ from $U_i$, $X_{i+2}$, $Y_{i+2}$, $Z_{i+2}$, $X_{i+3}$, $Y_{i+3}$, and $Z_{i+3}$.

(E) The plane $X_iY_iZ_{i+1}$ separates $Z_i$ from $Z_{i+2}$, $X_{i+3}$, $Y_{i+2}$ and $U_i$.

(F) The plane $Z_iZ_{i+2}U_i$ separates $X_i$, $Y_i$, $Z_{i+1}$, $Y_{i+1}$, $X_{i+1}$ from $X_{i+2}$, $Y_{i+2}$, $Z_{i+3}$, $Y_{i+3}$, $X_{i+3}$.

(G) The plane $Y_iZ_{i+2}U_i$ separates $X_i$ and $Z_{i+1}$ from $X_{i+3}$, $X_{i+2}$, $Y_{i+3}$, and $Z_{i+3}$.

(H) The plane $X_iZ_{i+2}U_i$ separates $Y_{i+2}$ and $Z_{i+1}$ from $Y_i$, $Z_i$, and $Z_{i+3}$.

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