AN UNSHELLABLE TRIANGULATION OF A TETRAHEDRON

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A triangulation K of a tetrahedron T is shellable if the tetrahedra $K_1, \cdots, K_n$ of K can be so ordered that $K_i \cup K_{i+1} \cup \cdots \cup K_n$ is homeomorphic to T for $i=1, \cdots, n$. Sanderson [Proc. Amer. Math. Soc. vol. 8 (1957) p. 917] has shown that, if K is a Euclidean triangulation of a tetrahedron then there is a subdivision K' of K which is shellable; and he raises the question of the existence of a Euclidean triangulation of a tetrahedron which is unshellable. Such a triangulation will be described here.

Let T be a tetrahedron each of whose edges has length 1.

We will describe a nontrivial Euclidean triangulation K of T such that, if R is any tetrahedron of K, then the closure of $(T-R)$ is not homeomorphic to T.

I. *Construction of K*: Let X_1, X_2, X_3, and X_4 be the vertices of T. The possible values for the letters i and j are 1, 2, 3, and 4 and addition involving i or j will be modulo 4.

For each i, let F_i denote the face of T opposite X_i, and let U_i be the midpoint of the interval X_iX_{i+2}. Observe that $U_1=U_3$ and $U_2=U_4$.

Let ϵ be the length of the shortest side of a triangle whose longest side is of length 1 and two of whose angles are 1° and 60°.

For each i, let Y_i denote the point of F_{i+1} at a distance $(3^{1/2}/2)\epsilon$ from X_i such that the angle $Y_iX_iX_{i+2}$ is 1°.

For each i, let Z_i denote the point of F_{i+2} such that the angle $Z_iX_iX_{i+3}$ is 1° and the angle $Z_iX_{i+1}X_i$ is 1°.

The fourteen vertices of our triangulation K are the points X_i, Y_i, Z_i, and U_i. It can be shown that no triangulation which has less than 14 vertices has the desired property.

The tetrahedra of our triangulation K are the tetrahedra of the forms:

(1) $X_iZ_iX_{i+1}Y_i$,
(2) $X_iZ_{i+1}X_{i+1}Y_i$,
(3) $Z_iZ_{i+1}X_{i+1}Y_i$,
(4) $Z_iZ_{i+1}X_{i+1}Y_{i+1}$,
(5) $Z_1Z_2Z_3Z_4$,
(6) $Z_iZ_{i+1}Y_iZ_{i+2}$,
(7) $X_iZ_{i+1}Y_iZ_{i+2}$,
(8) $X_iZ_{i+1}Y_{i+2}Z_{i+2}$,
(9) $X_iU_iY_{i+2}Z_{i+2}$,

(10) $X_iU_iY_iZ_{i+2}$,

(11) $Z_iU_iY_iZ_{i+2}$.

II. *Checking the construction*: The best method of doing this is to draw a big picture and label the vertices.

It is easy to check that for each tetrahedron R of K, the closure of $(T-R)$ is not homeomorphic to T.

In order to check that K is a triangulation, first observe that, for each i, the tetrahedra (1), (2), (3), and (4) fit together and form a thin rod having the triangles $X_iY_iZ_i$ and $X_{i+1}Y_{i+1}Z_{i+1}$ for its ends; the union of these rods forms a torus running along the edges X_iX_{i+1}. When (5) is added to this torus the remainder of T is divided into two congruent pieces each containing pieces of T along the faces F_i and F_{i+2}. After (6), (7), and (8) are added to the first five types there is only a small strip around X_iX_{i+2} remaining of T; (9) and (10) complete the faces of T and (11) fills in the final space.

To see that the tetrahedra all nest together properly in the order just described, the following facts will be useful. Fact A is needed for the "rods." Fact B is needed for (3). Facts C and D are needed as assurance that none of the tetrahedra of types (5) through (11) intersect the interior of the torus. Fact E is needed to show that (7) does not intersect either (2) or (6). And facts F, G, and H are needed to show that the tetrahedra of types (6) through (11) for $i=1$ do not intersect the tetrahedra of the same types for $i=3$. The facts can be easily proved using the definitions of ϵ, Y_i, and Z_i.

(A) The plane $X_iY_iZ_i$ separates X_{i+1}, Y_{i+1}, Z_{i+1} from X_{i-1}, Y_{i-1}, and Z_{i-1}.

(B) The points X_i and Y_i are on the same side of the plane $X_{i+1}Z_iZ_{i+1}$.

(C) The plane $Y_iZ_iZ_{i+3}$ separates X_i and X_{i+3} from U_i, X_{i+2}, Y_{i+2}, Z_{i+2}, X_{i+1}, Y_{i+1}, and Z_{i+1}.

(D) The plane $Y_iZ_iZ_{i+1}$ separates X_i and X_{i+1} from U_i, X_{i+2}, Y_{i+2}, Z_{i+2}, X_{i+3}, Y_{i+3}, and Z_{i+3}.

(E) The plane $X_iY_iZ_{i+1}$ separates Z_i from Z_{i+2}, X_{i+2}, Y_{i+2} and U_i.

(F) The plane $Z_iZ_{i+2}U_i$ separates X_i, Y_i, Z_{i+1}, Y_{i+1}, X_{i+1} from X_{i+2}, Y_{i+2}, Z_{i+3}, Y_{i+3}, X_{i+3}.

(G) The plane $Y_{i+2}Z_{i+2}U_i$ separates X_i and Z_{i+1} from X_{i+2}, X_{i+3}, Y_{i+3}, and Z_{i+3}.

(H) The plane $X_iZ_{i+2}U_i$ separates Y_{i+2} and Z_{i+1} from Y_i, Z_i, and Z_{i+3}.

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