The proofs of (4.2) and then of (1.1) are essentially the same as those of (3.2) and Dehn's lemma. The details are left to the reader.

References

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RESEARCH PROBLEMS

12. Richard Bellman: *Ordinary differential equations.*

It is known that if

a. A is a stability matrix, i.e., all characteristic roots have negative real parts,

b. $\|g(x)\|/\|x\|\to 0$ as $\|x\|\to 0$, $(\|x\|=\sum_i |x_i|)$,

then all solutions of $dx/dt=Ax+g(x)$ approach zero as $t\to\infty$, provided that $\|x(0)\|$ is sufficiently small (Poincaré-Lyapunov theorem).

If $x(0)=a_1c$, where c is a characteristic vector of A and a_1 is a scalar, what is the precise bound for $|a_1|$ in terms of A and $g(x)$? (Received January 7, 1958.)

13. Richard Bellman: *Partial differential equations.*

It is known that if $|g(u)|/|u|\to 0$ as $u\to 0$, then the solution of $u_t=u_{xx}+g(u)$, $u(0, t)=u(1, t)=0$, $t>0$, approaches zero as $t\to\infty$, provided that $\text{Max}_{0\leq x\leq 1} |u(x, 0)|$ is sufficiently small.

a. If $u(x, 0)=c_1$ what is the precise bound for $|c_1|$ in terms of $g(u)$?

b. If $u(x, 0)=c_1 \sin k\pi x$, what is the precise bound for $|c_1|$ in terms of $g(u)$?

14. Richard Bellman: *Functional equations.*

Let $f_n(u)$ be an analytic function of the function $u(x)$ and its first n derivatives $u'(x), \cdots, u^{(n)}(x)$, for $u\neq 0$, satisfying the functional equation

$$f_n(uv) = f_n(u) + f_n(v).$$

It is well-known that $f_0(u)=c_1 \log u$, and under much lighter conditions, and it is easy to show that $f_1(u)=c_1 \log u+c_2u'/u$.

What is the analytic form of f_n for general n? (Received January 9, 1958.)

15. Richard Bellman: *Functional equations and differential equations.*

Consider the nth order linear differential equation

$$\frac{d^n u}{dt^n} + a_1(t)\frac{d^{n-1}u}{dt^{n-1}} + \cdots + a_n(t)u = 0$$

and make the change of variable $u=vw$. The function w satisfies an equation of the same type with coefficients $b_i(t)$, $i=1, 2, \cdots, n$, where

$$b_1(t) = a_1(t) + nv'/v,$$

$$b_2(t) = a_2(t) + a_1(t)(n-1)v'/v + \frac{n(n-1)}{2} v''/v,$$

and so on.

Introduce the function

$$b_k(t) = f_k(a, v) = f_k(a_1, a_2, \cdots, a_k; v),$$

dependent upon v and its first k derivatives, for $k=1, 2, \cdots, n$. It is easy to see, from the origin of the coefficients b_k, that f_k satisfies the group property

$$f_k(a, v_1v_2) = f_k(f_1(a, v_1), f_2(a, v_1), \cdots, f_k(a, v_1); v_2).$$

What are the most general functions satisfying these functional relations? (Received January 9, 1958.)

16. Richard Bellman: *Functional equations and differential equations.*

If in the above linear equation, we introduce a change in the independent variable of the form $t=\phi(s)$, we obtain coefficients $b_i(t)$ which are functions of the a_i and the derivatives of ϕ. Similarly, the iterated substitution $t=\psi(\phi(s))$ gives rise to functional equations of the type given above. What are the most general functions satisfying these relations? (Received January 9, 1958)

17. Richard Bellman: *Matrix functional equations and differential equations.*

In the matrix differential equation $dX/dt=A(t)X(t)$ make the change of variable $X(t)=Y(t)Z(t)$. Then Z satisfies the equation

$$dZ/dt = (Y^{-1}A(t)Y - Y^{-1}Y')Z.$$

Introduce the matrix function

$$F(A; Y) = Y^{-1}AY - Y^{-1}Y'.$$

Then, as before,

$$F(A; Y_1Y_2) = F(F(A; Y_1), Y_2).$$

What is the most general matrix function of Y and Y' satisfying this equation? (Received January 9, 1958.)