ON THE NONEXISTENCE OF ELEMENTS OF HOPF INVARIANT ONE

BY J. F. ADAMS

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With the usual definitions of homotopy-theory, we have the following theorem.

THEOREM 1. (a) *S^{n-1} is not an H-space unless $n=2$, 4, or 8.*

(b) *There is no element of Hopf invariant one in $\pi_{2n-1}(S^n)$ unless $n=2$, 4, or 8.*

For the context of this question, see [**5**] (especially pp. 436–438), [**4**, Chapter VI] and [**6**, §§20, 21].

This theorem results from reasonings with secondary cohomology operations. It is generally understood that a secondary operation corresponds to a relation between primary operations. One may formalize the notion of a "relation" by introducing pairs (d, z), algebraic in nature, as follows.

Let p be a prime; let A be the Steenrod algebra [**2**, p. 43] over Z_p. One defines the notion of a graded left module M over the graded algebra A so that $M=\sum_q M_q$ and $A_qM_r\subset M_{q+r}$. For example, let us write $H^q(X)$ for $H^q(X; Z_p)$, $H^*(X)$ for $\sum_q H^q(X; Z_p)$ and $H^+(X)$ for $\sum_{q>0} H^q(X; Z_p)$; then $H^*(X)$ and $H^+(X)$ are graded left modules over A. Let M, N be such modules; one defines the notion of an A-map $f\colon M\to N$ of degree r so that $f(M_q)\subset N_{q+r}$.

A pair (d, z), then, is to have the following nature. The first entry d is to be an A-map $d\colon C_1\to C_0$ of degree zero. Here C_0, C_1 are to be modules in the above sense; we require, moreover, that they are locally finitely-generated and free, and that $(C_i)_q=0$ if $q<i$ $(i=0, 1)$. The second entry z is to be a homogeneous element of Ker d.

Let (d, z), then, be a pair of this sort. We call Φ a stable secondary cohomology operation associated with (d, z), if it satisfies the following axioms.

AXIOM (1). $\Phi(\epsilon)$ is defined for each A-map $\epsilon\colon C_0\to H^+(X)$ of degree $m\geqq 1$ and such that $\epsilon d=0$.

Such a map ϵ is determined by its values on the elements of an A-base of C_0. It therefore corresponds to a set of elements of $H^+(X)$. In particular, if C_0 is free on one given generator c, we write $u=\epsilon c$; we may thus consider Φ as a function of one variable u, where u runs over a subset of $H^+(X)$. In this case we write $\Phi(u)$ for $\Phi(\epsilon)$.

For the next axiom, set $\deg(z)=n+1$, let $f\colon C_1\to H^+(X)$ run over the A-maps of degree $(m-1)$, and let $Q^{m+n}(d, z; X)$ be the set of elements of the form fz.

Axiom (2). $\Phi(\epsilon) \in H^{m+n}(X)/Q^{m+n}(d, z; X)$.

For the next axiom, let g: $Y \to X$ be a map.

Axiom (3). $g^*\Phi(\epsilon) = \Phi(g^*\epsilon)$.

For the next axiom, let (X, Y) be a pair, and let ϵ: $C_0 \to H^+(X)$ be a map of degree $m \geqq 1$ such that $\epsilon d = 0$ and $i^*\epsilon = 0$. We can now form the following diagram.

$$\begin{array}{ccccccccc} H^+(Y) & \xleftarrow{i^*} & H^+(X) & \xleftarrow{j^*} & H^+(X, Y) & \xleftarrow{\delta} & H^+(Y) & \xleftarrow{i^*} & H^+(X) \\ & & & \overset{\epsilon}{\nwarrow} & \uparrow \eta & & \uparrow \zeta & & \\ & & & & C_0 & \xleftarrow{(-1)^m d} & C_1 & & \end{array}$$

Axiom (4). $i^*\Phi(\epsilon) = \{\zeta z\} \bmod i^*Q^{m+n}(d, z; X)$.

For the next axiom, let SX be the suspension of X, and let $\sigma: H^+(X) \to H^+(SX)$ be the suspension isomorphism. Let ϵ be as above.

Axiom (5). $\sigma\Phi(\epsilon) = \Phi(\sigma\epsilon)$.

Theorem 2. *Given any pair* (d, z) (*as above*), *there is at least one stable secondary cohomology operation* Φ *associated with it* (*in the sense of the axioms above*).

This theorem is proved by the method of the universal example. The next theorem allows us to study the operations Φ by applying homological algebra (see [3]) to the pairs (d, z).

Theorem 3. (a) *If* Φ, Φ' *are two operations associated with the same pair* (d, z) *then there is an element* c *in* $(C_0/dC_1)_n$ *such that*

$$\Phi(\epsilon) - \Phi'(\epsilon) = \{\epsilon c\}.$$

(b) *Suppose given* d (*as above*), *elements* z_t *in* Ker d, *and operations* Φ_t *associated with the pairs* (d, z_t). *Suppose* $z = \sum_t a_t z_t$ $(a_t \in A)$. *Then there is an operation* Φ *associated with* (d, z) *such that*

$$\sum_t a_t\Phi_t(\epsilon) = \{\Phi(\epsilon)\} \bmod \sum_t a_t Q^{m+n_t}(d, z_t; X).$$

(c) *Suppose given a diagram*

$$\begin{array}{ccc} C_1 & \xrightarrow{m_1} & C_1' \\ d \downarrow & & \downarrow d' \\ C_0 & \xrightarrow{m_0} & C_0' \end{array}$$

in which d, d' are as above, and m_0, m_1 are A-maps of degree zero. Let Φ be an operation associated with a pair (d, z). Then there is an operation Φ' associated with $(d', m_1 z)$ such that

$$\Phi(\epsilon' m_0) = \{\Phi'(\epsilon')\}$$

for each ϵ': $C_0' \to H^+(X)$ of the sort considered above.

One may show that for operations in one variable, there is a Cartan formula for expanding $\Phi(uv)$, where uv is a cup-product.

We now take $p=2$. We also omit to summarize some work with homological algebra. This work leads us to consider certain pairs (d, z). By applying Theorem 2, we obtain secondary operations $\Phi_{i,j}(u)$ for $0 \leqq i \leqq j$, $j \neq i+1$. The operation $\Phi_{j,i}(u)$ is of degree $2^i + 2^j - 1$, and it is defined on classes u such that $Sq^{2^r}(u) = 0$ for $0 \leqq r \leqq j$.

Let P be complex projective space of infinitely-many dimensions, and let y be a generator of $H^2(P)$ (by which we mean $H^2(P; Z_2)$). We may ask for the values of the operations $\Phi_{i,j}$ in $H^*(P)$. Now, $\Phi_{i,j}(y^r)$ is defined only if $r \equiv 0$ and mod 2^j. Moreover, the degree of $\Phi_{i,j}$ is odd unless $i=0$ and $j>0$; so that $\Phi_{i,j}(y^r)$ lies in a zero group unless $i=0$ and $j>0$. It remains only to consider $\Phi_{0,j}(y^{n2^j})$; this is defined modulo zero.

THEOREM 4.

$$\Phi_{0,j}(y^{n2^j}) = ny^{(n+1/2)2^j} \qquad \text{(mod zero)}.$$

In the proof of this theorem we make essential use of a formula for the composite operation $\Phi_{0,j} Sq^{2^j}$. This formula is proved by applying Theorem 3.

THEOREM 5. *For each $k \geqq 3$ we have a formula*

$$\sum_{i,j;j\leq k} a_{i,j,k} \Phi_{i,j}(u) = Sq^{2^{k+1}}(u) \qquad (\text{mod } Q).$$

The formula is valid on classes u such that $Sq^{2^r}(u) = 0$ for $0 \leqq r \leqq k$, and holds modulo a certain subgroup Q. It is proved as follows. By applying Theorem 3, we obtain a formula

$$\sum_{i,j;j\leq k} a_{i,j,k} \Phi_{i,j}(u) = \lambda Sq^{2^{k+1}}(u) \qquad (\text{mod } Q)$$

in which $a_{i,j,k} \in A$, and the coefficient λ remains to be determined. Applying the formula to a suitable class u in $H^*(P)$, we determine $\lambda = 1$.

To prove Theorem 1, it is sufficient to prove it for the case $n = 2^m$. This case follows immediately from Theorem 5, using the same argument as that used by Adem [1, §4] in the case $n \neq 2^m$.

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