

# ON RAPIDLY MIXING TRANSFORMATIONS AND AN APPLICATION TO CONTINUED FRACTIONS

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1. Let  $\Omega$  be a measure space (of total measure 1) and let  $T$  be a measure preserving transformation in the sense that

$$(1.1) \quad \mu\{T^{-1}(A)\} = \mu\{A\},$$

where  $T^{-1}(A)$  denotes the inverse image of  $A$  (we do not assume that  $T$  is necessarily one-to-one).

Let

$$(1.2) \quad V(\omega) = \begin{cases} 1, & \omega \in B, \\ 0, & \omega \notin B, \end{cases}$$

and consider

$$(1.3) \quad \sum_{k=1}^n V(T^k\omega).$$

It will be our purpose to sketch a proof of the theorem that under a suitable condition on  $T$  the "conditional" measure

$$(1.4) \quad \frac{\left\{ \frac{\sum_{k=1}^n V(T^k\omega)}{n} - \mu\{B\} \right\}}{\mu\{B\}} < \frac{\alpha}{n^{1/2}}, \quad \omega \in B$$

approaches as  $n$  tends to infinity

$$(1.5) \quad \frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\alpha} e^{-u^2/(2\sigma^2)} du,$$

where  $\sigma$  is, in general, not known explicitly.

The condition to be imposed on  $T$  is that of "exponentially rapid mixing" and can be stated as follows:

If  $\nu$  and  $\nu_m$  are defined by

$$\nu\{A\} = \frac{\mu\{A \cap B\}}{\mu\{B\}}, \quad \nu_m\{A\} = \frac{\mu\{A \cap B \cap T^{-m}B\}}{\mu\{B \cap T^{-m}B\}}$$

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then for every measurable set  $A$

$$(1.6) \quad \begin{aligned} |\nu\{T^{-k}A\} - \mu\{T^{-k}A\}| &= |\nu\{T^{-k}A\} - \mu\{A\}| \leq H e^{-\epsilon k} \mu\{A\}, \\ |\nu_m\{T^{-k}A\} - \mu\{A\}| &\leq H_m e^{-\epsilon k} \mu\{A\}. \end{aligned}$$

$\epsilon > 0$  is an absolute constant, but  $H_m$  may vary with  $m$ .

This condition may seem more severe than Bernstein's (Math. Ann. vol. 97 (1927) pp. 1-59), but Bernstein's conditions would ask for a "uniformly rapid" decrease of all expressions

$$\left| \frac{\mu\{T^{-k}A \cap B_1 \cap \dots \cap B_m\}}{\mu\{B_1 \cap \dots \cap B_m\}} - \mu\{A\} \right|$$

where each  $B_i = B$  or  $\Omega - B$ .

2. Let  $\nu$  be defined as follows:

$$(2.1) \quad \nu(A) = \frac{\mu(A \cap B)}{\mu(B)},$$

where  $B$  is a fixed set of positive measure. We set

$$(2.2) \quad {}_{(k)}P_n^{(1)} = \frac{\int_{\Omega} V(T^k\omega) \exp\left(-u \sum_{r=k+1}^{k+n} V(T^r\omega)\right) d\nu}{\int_{\Omega} V(T^k\omega) d\nu},$$

$$(2.3) \quad {}_{(k)}P_n^{(2)} = \frac{\int_{\Omega} (1 - V(T^k\omega)) \exp\left(-u \sum_{r=k+1}^{k+n} V(T^r\omega)\right) d\nu}{\int_{\Omega} (1 - V(T^k\omega)) d\nu}$$

(with the obvious convention  $P_0 \equiv 1$ ) and

$$(2.4) \quad S_k^{(q)} = \sum_{n=0}^{\infty} {}_{(k)}P_n^{(q)} z^n, \quad q = 1, 2.$$

Setting also

$$B^{(1)} = B, \quad B^{(2)} = \Omega - B \quad V^{(1)} = 1, \quad V^{(2)} = 0$$

we verify by an immediate calculation that

$$(2.5) \quad \sum_q \nu\{T^{-k}(B^{(q)})\} S_k^{(q)} = 1 + z \sum_q e^{-uV^{(q)}} \nu\{T^{-(k+1)}(B^{(q)})\} S_{k+1}^{(q)},$$

where  $\nu\{T^{-k}(B^{(a)})\}S_k^{(a)}$  is taken to be zero whenever  $\nu\{T^{-k}(B^{(a)})\}=0$ .

Multiplying both sides of (2.5) by  $z^k$  and summing on  $k$  from 0 to  $\infty$  we obtain

$$(2.6) \quad S_0^{(1)} = \frac{1}{1-z} - (1-e^{-u}) \sum_{k=1}^{\infty} \nu\{T^{-k}(B^{(1)})\} S_k^{(1)} z^k.$$

It should, of course, be remembered that  $S$  depends on  $z$  and  $u$ . Dropping the superscript 1, we get, setting

$$(2.7) \quad Q(z) = \sum_{k=1}^{\infty} \nu\{T^{-k}(B)\} (S_k - S_0) z^k,$$

$$(2.8) \quad S_0 = \frac{1}{(1-z) \left[ 1 + (1-e^{-u}) \sum_{k=1}^{\infty} \nu\{T^{-k}(B)\} z^k \right]} - \frac{(1-z)(1-e^{-u})Q(z)}{(1-z) \left[ 1 + (1-e^{-u}) \sum_{k=1}^{\infty} \nu\{T^{-k}(B)\} z^k \right]}.$$

3. Let us expand  $(1-e^{-u})(1-z)Q(z)$  in a power series

$$(3.1) \quad (1-e^{-u})(1-z)Q(z) = \sum_{r=1}^{\infty} \gamma_r(u) z^r$$

(recall that (2.7) is not a power series expansion of  $Q$  since the  $S$ 's themselves depend on  $z$ ) and let us also write

$$(3.2) \quad \frac{1}{(1-z) \left[ 1 + (1-e^{-u}) \sum_{k=1}^{\infty} \nu\{T^{-k}(B)\} z^k \right]} = \sum_{r=0}^{\infty} \beta_r(u) z^r.$$

From (2.8) we have comparing coefficients

$$(3.3) \quad \begin{aligned} {}_{(0)}P_n &= \frac{\int_B \exp\left(-u \sum_{r=1}^n V(T^r\omega)\right) d\nu}{\int_B d\nu} \\ &= \frac{\int_B \exp\left(-u \sum_{r=1}^n V(T^r\omega)\right) d\mu}{\mu\{B\}} = \beta_n(u) + \sum_{r=1}^n \gamma_r(u) \beta_{n-r}(u), \end{aligned}$$

and hence setting  $u = -i\xi/n^{1/2}$ ,

$$\begin{aligned}
 & \frac{\int_B \exp\left(\frac{i\xi}{n^{1/2}} \left[ \sum_1^n V(T^r\omega) - n\mu\{B\} \right]\right) d\mu}{\mu\{B\}} \\
 (3.4) \quad &= e^{-i\xi\mu\{B\}n^{1/2}} \beta_n\left(-\frac{i\xi}{n^{1/2}}\right) \\
 &+ e^{-i\xi\mu\{B\}n^{1/2}} \sum_{r=1}^n \gamma_r\left(-\frac{i\xi}{n^{1/2}}\right) \beta_{n-r}\left(-\frac{i\xi}{n^{1/2}}\right).
 \end{aligned}$$

4. From condition (1.6) it follows immediately that

$$(4.1) \quad \sum_{k=1}^\infty \nu\{T^{-k}(B)\} z^k = \frac{\mu\{B\}z}{1-z} + R(z),$$

where  $R(z)$  is analytic for  $|z| < 1 + \epsilon$ .

It is now quite easily shown that

$$(4.2) \quad \lim_{n \rightarrow \infty} e^{-i\xi\mu\{B\}n^{1/2}} \beta_n\left(-\frac{i\xi}{n^{1/2}}\right) = e^{-\sigma^2\xi^2/2},$$

where

$$(4.3) \quad \sigma^2 = \mu\{B\} [1 - \mu\{B\}] + 2\mu\{B\} R(1).$$

It is harder to prove that

$$(4.4) \quad \lim_{n \rightarrow \infty} n \sup_{1 \leq r \leq n} \left| \gamma_r\left(-\frac{i\xi}{n^{1/2}}\right) \right| = 0,$$

but once this is done one gets from (3.4) and (4.2) that

$$(4.5) \quad \lim_{n \rightarrow \infty} \frac{\int_B \exp\left(\frac{i\xi}{n^{1/2}} \left[ \sum_1^n V(T^k\omega) - n\mu\{B\} \right]\right) d\mu}{\mu\{B\}} = e^{-\sigma^2\xi^2/2}$$

and the theorem announced in §1 follows.

It should be pointed out that in proving (4.4) one has to apply (1.6) to all  $\nu_m$ .

5. The general theorem of this note was motivated by an application to continued fractions.

In this case  $\Omega$  is the interval  $[0, 1]$  and the invariant measure

$$(5.1) \quad \mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}.$$

The transformation  $Tx$  in question is given by the formula

$$(5.2) \quad Tx = \frac{1}{x} - \left[ \frac{1}{x} \right]$$

and the crucial property (1.6) was proved by Paul Lévy [1].

Our result is thus that *the number of times a specified digit occurs among the first  $n$  digits in a continued fraction is asymptotically normally distributed.*

The method used in proving our main result was suggested by the occupation time problem for Markoff chains. The fuller discussion of this connection as well as that of a number of related results will appear elsewhere.

#### REFERENCES

1. Paul Lévy, *Theorie de l'addition des variables aleatoires*, Paris, Gauthier-Villars, 1937, page 298 ff.

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