

REPRESENTATION OF ABSTRACT RIESZ POTENTIALS OF THE ELLIPTIC TYPE

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Using semigroup theory we are able to obtain an abstract definition of the Riesz potentials of the elliptic type as closed linear operators, as well as a representation for them without using continuation.

Let

$$T(\xi), \xi = (\xi_1, \xi_2, \dots, \xi_n), -\infty < \xi_k < \infty,$$

be an n -parameter group (strongly continuous) of endomorphisms over a B -space X . Let

$$T(\xi) = \prod_{k=1}^n T_k(\xi_k),$$

each $T_k(\xi_k)$ being a strongly continuous one-parameter group with infinitesimal generator A_k . Let

$$C = \sum_{i=1}^n \sum_{j=1}^n a_{ij} A_i A_j,$$

where the matrix $[a_{ij}]$ is real symmetric and positive definite. Then C is the infinitesimal generator of a one-parameter strongly continuous semigroup $S(t)$, $0 < t$, and further $\sup \|S(t)\| < \infty$. Now, in some previous work the author has shown that in such a case it is possible to define $(-C)^\alpha$, $\text{Re } \alpha > 0$, as closed linear operators, interpolating integral powers and having the semigroup property in α . Moreover, they have an explicit representation as Bochner integrals in terms of $S(t)$, which for $0 < \text{Re } \alpha < 1$, is

$$(1) \quad (-C)^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty [S(t)x - x] t^{-\alpha-1} dt,$$

for $x \in D(C)$. Next, to simplify the notation, let

$$C = \sum_{i=1}^n A_i^2.$$

Then for every $x \in X$,

$$(2) \quad S(t)x = \frac{1}{(2(\pi t)^{1/2})^n} \int_{E_n} T(\xi)x \exp \left[- \sum_1^n \xi_k^2 / 4t \right] d\xi_1 \cdots d\xi_n.$$

The Riesz potentials are obtained by substituting (2) into (1). Thus:

$$(-C)^{\alpha}x = \frac{4^{\alpha}\Gamma(\alpha + n/2)}{\pi^{n/2}\Gamma(-\alpha)} \int_{E_n^+} [T(\xi)x + T(-\xi)x - 2x] |\xi|^{-2\alpha-n} dV_{\xi}$$

where

E_n is the 2^n -ant in which $\xi_k \geq 0$ for all k ,

$$|\xi| = [\xi_1^2 + \dots + \xi_n^2]^{1/2},$$

dV_{ξ} is the n -dimensional volume element.

With slight modification it is possible to write this as:

$$(-C)^{\alpha}x = \frac{4^{\alpha}\Gamma(\alpha + n/2)}{\pi^{n/2}\Gamma(-\alpha)(n + 2\alpha - 1)} \int_{E_n^+} [T(\xi) - T(-\xi)] \cdot \left[\sum_1^n \xi_k A_k x \right] |\xi|^{-2\alpha-n} dV_{\xi},$$

where the integral is to be taken in the Cauchy sense at infinity. In this latter form, for $\alpha = 1/2$ for instance, we get an abstract version of the conjugate transform in $L_p(E_n)$ spaces in its usual representation.

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