BOOK REVIEWS

Introduction to Riemann surfaces. By George Springer. Reading, Addison-Wesley, 1957. 8+307 pp. $9.50.

Of all requests for bibliographical information made to the reviewer none have come as frequently as that for "a good modern introduction to Riemann surfaces" from those not specialists in Function Theory. This request has been difficult to answer, for while recent books of Nevanlinna, Schiffer-Spencer and Pfluger contain introductory material to a greater or lesser extent what has been needed is a book which is avowedly a textbook on the subject. This is the need which the present book by Springer aims to fill.

The subject in hand is without doubt one of the hardest in which to write an effective text. The reason for this is the fact that, just as from the study of Riemann surfaces has developed a large part of modern Mathematical endeavour, so now in order to present the theory in its proper context it is necessary to call on many branches of Mathematics. The writer is faced at every step with difficult choices as to what to assume and what to develop from first principles. Let it be said at once that on the whole the author has done an excellent job. In a subject as well developed as the present it would indeed be hard to display much originality in the actual content of the proofs and those familiar with the sources will recognize many of those given here. Nevertheless the author has built up the logical structure carefully, blended the proofs skillfully to provide good unity of style and for the most part has smoothed the passage from one concept to another with carefully thought out motivation.

We will now describe the actual contents of the book.

Chapter 1 consists of an introduction, for the most part heuristic, to the theory. Starting with the simplest notions of algebraic functions and their integrals and the associated Riemann surfaces, the author discusses some geometric-topological aspects of the latter. He then passes on to a discussion of fluid flows and potentials, first in the plane, then on differential-geometric surfaces. He exhibits the nature of the simplest singularities and connects them with meromorphic functions. Finally he goes into somewhat more detail in the case of the torus.

Chapter 2 contains an introduction to the simplest concepts of point set topology. There follows a discussion of (two-dimensional) manifolds, including Prüfer's example of such not possessing a countable base. The chapter concludes with the introduction of the concept of an (abstract) Riemann surface.

Chapter 3, dealing with the Riemann surface of an analytic function, contains much of the work usually treated in a study of analytic continuation.

Chapter 4 discusses covering manifolds with special emphasis on the universal covering manifold, homotopy, the fundamental group, simple connectivity and covering transformations.

Chapter 5 contains a treatment of combinatorial topology in the special case of a triangulable manifold (surface) with a special discussion of orientability. The normal forms of compact orientable surfaces are given. The homology groups (integer coefficients) are defined and their topological invariance proved in this special case. This is trivial except for dimension one where it is proved by showing isomorphism with the abelianized fundamental group. Finally the homology properties of compact surfaces are given.

Chapter 6 deals with first and second order differentials and their integrals. Use is made of partitions of unity. Stokes' theorem is proved and the exterior differential notation introduced. Harmonic and analytic differentials are defined and some simple consequences drawn.

Chapter 7 provides a brief introduction to Hilbert space. It is shown that the first order differentials form a Hilbert space in a natural manner. A discussion of smoothing operators leads up to a proof of Weyl's lemma and the derivation of various decompositions for differentials.

In Chapter 8 is proved the existence of various harmonic and analytic differentials with specified singularities on Riemann surfaces (some of them applying only in the compact case). It is shown that every Riemann surface has a countable base.

Chapter 9 begins with Koebe's proof of the existence of a conformal mapping of a schlichtartig Riemann surface onto a plane domain. This is followed by its application to the conformal mapping of the universal covering surface of a Riemann surface onto a surface of one of the three canonical types. This leads to a discussion of automorphic functions, the corresponding discontinuous groups and their fundamental regions. It is proved that every Riemann surface is triangulable. Then the group of self-conformal mappings of a Riemann surface is studied briefly and those surfaces for which this group is not discontinuous enumerated.

Finally in Chapter 10 we find the classical theory for functions and differentials on a closed Riemann surface. The standard topics are presented: the linear space of regular differentials, Riemann's bilinear relations, normalized differentials of the first, second and third kinds, divisors, the Riemann-Roch theorem, the Weierstrass gap theorem, Weierstrass points, Abel's theorem, the Jacobi inversion problem. It is shown that the meromorphic functions on a compact surface form an algebraic function field. The hyperelliptic case is discussed in detail as an example.

The instructor employing this book will naturally be led to speculate on the author's choice of basic material from other fields. What is required in addition to a standard rigorous course in complex variable comprises results from group theory, Lebesgue integration theory, point set topology, combinatorial topology and Hilbert space. The author chooses to presuppose the first two and develop the latter three from first principles. A good case can be made for these choices. In the treatment of combinatorial topology it is possible to work with a very simple special case. The amount of Hilbert space theory is so small that its omission would save little space at a considerable loss of elegance. On the other hand the group theory employed will most likely be known to all students on the level at which this book will be used. However there seems to be no reason to assume that by and large such students will know Lebesgue integration theory rather than the elements of point set topology. Of course a detailed treatment of the necessary results from the former would require much more space than that devoted by the author to the latter. A case could be made, though, for just giving a brief summary of the concepts and results borrowed from both these topics.

The omission of certain material on a more sophisticated level seems regrettable to the reviewer. The classical theorem of Brill and Noether is relegated to an exercise (where an unfortunate misprint occurs in its statement). To be sure this result is easily derived but this means also that it would cost little to make a bow to tradition here. A more serious deficiency is the fact that the important concept of the double of a finite Riemann surface with boundary appears only in two problems. Despite the importance and usefulness of this notion the reviewer knows of no readily accessible detailed expository treatment of it and one would have been welcome. With the author going as far as he does in developing the Koebe slit mapping for schlichtartig surfaces it would have required little additional effort to develop some further properties of the minimal slit mapping. Finally no mention at all is made of quadratic differentials which play a key role in the most sophisticated geometrical developments in Function Theory. It would have been a simple matter to give a short account of the basic algebraic results for them as was done by

Hensel and Landsberg even at a time when they represented only a formal generalization.

As in any book written by mortal man there are a number of rough places. Although the reviewer did not read every proof in detail he observed the following. No attempt is made to motivate the use of a definition of "schlichtartig" not quite the usual one. The definition of the mapping h in the middle of p. 136 is either confused or confusing. Not sufficient discussion is given of the distinctions between the various decompositions of differentials in Chapter 7. On p. 237 it is not made clear what is meant by "adjacent sides". However without greater precision the last two sentences in the paragraph following Theorem 9–12 are questionable. Consider the modular group. On p. 268 there seems to be slight verbal confusion between divisors which are integral and those equivalent to an integral divisor. There is little attempt made to motivate the Jacobi inversion problem. Also in its discussion we find on p. 281 $P_1, \cdots, P_n$ specified as distinct points but at the top of p. 284 the conclusions are applied without further ado to the specialization $P_1 \cdots P_g = P_0^g$. On p. 295 in the proof that a certain surface is not hyperelliptic it should be observed that the powers of z do not form a subfield (it should be the rational functions of z). Most of these points are comparatively minor and easily rectified but might distract the conscientious student. Finally a small number of misprints, pure and simple, were observed.

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Teoria miary i calki Lebesgue'a. (Polish). By S. Hartman and J. Mikusinski. Panstwowe Wydawnictwo Naukowe, Warszawa, 1957. 140 pp. zl. 10.

This book is a short textbook on the theory of measure and of the Lebesgue integral, containing the classical material of the subject which corresponds to the requirements of the curriculum in Polish universities.

The main purpose of the book is to present that part of measure theory which has shown itself to be most useful in its applications in other fields such as the theory of probability and theoretical physics.

There are twelve chapters in the book. 1. Introductory concepts; 2. Lebesgue's measure of linear sets; 3. Measurable functions; 4. The Lebesgue definite integral; 5. Convergence in measure; 6. Integration and differentiation. Functions of bounded variation; 7. Absolutely continuous functions; 8. L^p spaces; 9. Orthogonal expansions; 10. Measure in plane and in space; 11. Multiple integrals; 12. The Stieltjes integral.