In this paper we combine the methods of Aronszajn and Milgram [3] with those previously employed by the author [9] and solve very general boundary value problems for elliptic equations. For convenience we consider equations, but much of what we say can be carried over to systems without difficulty.

Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) be a sequence of indices, each \( \geq 0 \), and set

\[
|\mu| = \sum \mu_k, \quad \xi^\mu = \xi_1^{\mu_1} \xi_2^{\mu_2} \cdots \xi_n^{\mu_n},
\]

\[
D^\mu = \partial |^{|\mu|} / (i \partial x_1)^{\mu_1} (i \partial x_2)^{\mu_2} \cdots (i \partial x_n)^{\mu_n}
\]

where \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \) is any \( n \)-dimensional vector. The linear partial differential operator of order \( 2r \)

\[
A = \sum_{|\mu|=2r} a_\mu(x) D^\mu
\]

with complex coefficients \( a_\mu(x) \) is elliptic in a region \( R \subseteq \mathbb{R}^n \) if its characteristic polynomial

\[
P(x, \xi) = \sum_{|\mu|=2r} a_\mu(x) \xi^\mu
\]

does not vanish in \( R \) for real \( \xi \neq 0 \). If \( R \) is the closure \( \bar{G} \) of a bounded domain \( G \), we shall say that \( A \) is properly elliptic in \( \bar{G} \) if in addition it satisfies at every point \( x \) on the boundary \( \partial G \) of \( G \) (cf. [2; 5; 8]).

**Condition 1.** For every real vector \( \tau \neq 0 \) parallel to \( \partial G \) at \( x \) and every real \( \nu \neq 0 \) normal to \( \partial G \) at \( x \), the polynomial \( P(z) = P(x, \tau + \nu z) \) has exactly \( r \) roots \( \lambda_k(\tau, \nu), k = 1, 2, \ldots, r \), with positive imaginary parts.

If \( n > 2 \), all elliptic operators are properly elliptic.

By a boundary operator we shall mean a linear partial differential operator whose coefficients need merely be defined on \( \partial G \). If

\[
B_j = \sum_{|\mu| \leq m_j} b_{\mu j}(x) D^\mu, \quad j = 1, 2, \ldots, r,
\]

is a set of such operators, we set

\[
Q_j(x, \xi) = \sum_{|\mu| = m_j} b_{\mu j}(x) \xi^\mu, \quad j = 1, 2, \ldots, r.
\]

We shall say that the set \( \{ B_j \}_{j=1}^r \) "covers" a properly elliptic operator \( A \) if each \( m_j < 2r \) and the \( B_j \) satisfy at each point \( x \in \partial G \).
CONDITION 2. For every $r \neq 0$ parallel to $\mathcal{G}$ at $x$ and every $v \neq 0$ normal to $\mathcal{G}$ at $x$ the polynomials $Q_j(x) = Q_j(x, \tau + zv), j = 1, 2, \ldots, r,$ are linearly independent modulo the polynomial

$$S(z) = \prod_{k=1}^{r} (z - \lambda_k(r, v)).$$

From now on we shall assume that $G$ is bounded, $\mathcal{G}$ is of class $C^\infty$, and all functions considered are in $C^\infty(\mathcal{G})$. We set

$$(u, v)_s = \int_{\mathcal{G}} \sum_{|\alpha| \leq s} D^\alpha u D^\alpha v dx$$

and denote the formal adjoint of $A$ by $A^*$. The set $\{B_j\}_{j=1}^{r}$ is called normal if $m_j \neq m_k$ for $j \neq k$ and $Q_j(x, v) \neq 0$ for each $j$ and $x \in \mathcal{G}$ when $v \neq 0$ is normal to $\mathcal{G}$ at $x$.

**Lemma 1** (Aronszajn-Milgram [3]). Let $A$ be an elliptic operator of order $2r$, and let $\{B_j\}_{j=1}^{r}$ be a normal set of boundary operators of orders $m_j \leq 2r$, respectively. Then we can find another normal set $\{B_j^*\}_{j=1}^{r}$ such that

$$(1)\quad (u, A^* v) = (Au, v)$$

for all $v$ satisfying

$$(2)\quad B_j^* v = 0 \quad \text{on } \mathcal{G}, \quad j = 1, 2, \ldots, r,$$

if and only if

$$(3)\quad B_j u = 0 \quad \text{on } \mathcal{G}, \quad j = 1, 2, \ldots, r.$$  

We call the set $\{B_j^*\}_{j=1}^{r}$ adjoint to $\{B_j\}_{j=1}^{r}$ relative to $A$. We also have

**Lemma 2.** The normal set $\{B_j\}_{j=1}^{r}$ covers $A$ if and only if every normal set adjoint to $\{B_j\}_{j=1}^{r}$ relative to $A$ also covers $A$.

The boundary value problem $\pi(A, f, u_0, B_j)$ is to find a function $u \in C^\infty(\mathcal{G})$ such that $Au = f$ in $\mathcal{G}$ and $B_j u = B_j u_0$ on $\mathcal{G}, j = 1, 2, \ldots, r$. We now state our main result.

**Theorem.** Let $A$ be a properly elliptic operator of order $2r$ which is covered by a normal set $\{B_j\}_{j=1}^{r}$. Let $\{B_j^*\}_{j=1}^{r}$ be any normal set adjoint to $\{B_j\}_{j=1}^{r}$ relative to $A$. Then the boundary value problem $\pi(A, f, u_0, B_j)$ has a solution for every $f$ and $u_0$ if and only if the solution of $\pi(A^*, 0, 0, B_j^*)$ is unique.
OUTLINE OF PROOF. Complete $C^a(\bar{G})$ with respect to the norm $\| \cdot \|_{2r}$ and call the resulting Hilbert space $H$. For convenience we assume $u_0 = 0$. Set

$$[u, v] = (A^* u, A^* v) + \sum_{j=1}^r \langle B_j^* u, B_j^* v \rangle_{2r-m_j},$$

where the $\langle \cdot, \cdot \rangle_{2r}$ are appropriate boundary inner products (cf. [1; 2; 8]). By Lemma 2 it follows that the set $\{B_j^*\}_{j=1}^r$ covers $A$ (and consequently $A^*$). The results of [1; 2; 8] and the uniqueness of $\pi(A^*, 0, 0, B_j^*)$ imply

$$c^{-1} \| v \|_{2r}^2 \leq [v, v] \leq c \| v \|_{2r}^2$$

for all $v \in H$ such that

$$[g, v] = (f, v)$$

for all $v \in H$.

Applying regularity theory similar to that of Nirenberg [7] and Browder [4] we then prove that $g \in C^a(\bar{G})$. Hence $u = A^* g$ satisfies $Au = f$ in $G$ and $(u, A^* v) = (Au, v)$ for all $v \in C^a(\bar{G})$ satisfying (2). Thus $u$ satisfies (3) by Lemma 1 and the proof is complete.

REFERENCES


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