GENERAL BOUNDARY VALUE PROBLEMS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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In this paper we combine the methods of Aronszajn and Milgram [3] with those previously employed by the author [9] and solve very general boundary value problems for elliptic equations. For convenience we consider equations, but much of what we say can be carried over to systems without difficulty.

Let $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ be a sequence of indices, each $\geqq 0$, and set

$$|\mu| = \sum \mu_k, \qquad \xi^\mu = \xi_1^{\mu_1}\xi_2^{\mu_2} \cdots \xi_n^{\mu_n},$$
$$D^\mu = \partial^{|\mu|}/(i\partial x_1)^{\mu_1}(i\partial x_2)^{\mu_2} \cdots (i\partial \chi_n)^{\mu_n}$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ is any n-dimensional vector. The linear partial differential operator of order $2r$

$$A = \sum_{|\mu| \leq 2r} a_\mu(x) D^\mu$$

with complex coefficients $a_\mu(x)$ is elliptic in a region $R \subseteq E^n$ if its characteristic polynomial

$$P(x, \xi) = \sum_{|\mu| = 2r} a_\mu(x)\xi^\mu$$

does not vanish in R for real $\xi \neq 0$. If R is the closure $\bar{G}$ of a bounded domain G, we shall say that A is properly elliptic in $\bar{G}$ if in addition it satisfies at every point x on the boundary $\dot{G}$ of G (cf. [2; 5; 8]).

CONDITION 1. For every real vector $\tau \neq 0$ parallel to $\dot{G}$ at x and every real $\nu \neq 0$ normal to $\dot{G}$ at x, the polynomial $P(z) = P(x, \tau + z\nu)$ has exactly r roots $\lambda_k(\tau, \nu)$, $k = 1, 2, \cdots, r$, with positive imaginary parts.

If $n > 2$, all elliptic operators are properly elliptic.

By a boundary operator we shall mean a linear partial differential operator whose coefficients need merely be defined on $\dot{G}$. If

$$B_j = \sum_{|\mu| \leq m_j} b_{j\mu}(x) D^\mu, \qquad j = 1, 2, \cdots, r,$$

is a set of such operators, we set

$$Q_j(x, \xi) = \sum_{|\mu| = m_j} b_{j\mu}(x)\xi^\mu, \qquad j = 1, 2, \cdots, r.$$

We shall say that the set $\{B_j\}_{j=1}^r$ "covers" a properly elliptic operator A if each $m_j < 2r$ and the B_j satisfy at each point $x \in \dot{G}$.

CONDITION 2. For every $\tau \neq 0$ parallel to $\dot{G}$ at x and every $\nu \neq 0$ normal to $\dot{G}$ at x the polynomials $Q_j(z) = Q_j(x, \tau + z\nu), j = 1, 2, \cdots, r$, are linearly independent modulo the polynomial

$$S(z) = \prod_{k=1}^{r} (z - \lambda_k(\tau, \nu)).$$

From now on we shall assume that G is bounded, $\dot{G}$ is of class C^∞, and all functions considered are in $C^\infty(\overline{G})$. We set

$$(u, v)_s = \int_G \sum_{|\mu| \leq s} D^\mu u \overline{D^\mu v} dx$$

$$\|v\|_s^2 = (v, v)_s, \qquad (u, v) = (u, v)_0$$

and denote the formal adjoint of A by A^*. The set $\{B_j\}_{j=1}^r$ is called *normal* if $m_j \neq m_k$ for $j \neq k$ and $Q_j(x, \nu) \neq 0$ for each j and $x \in \dot{G}$ when $\nu \neq 0$ is normal to $\dot{G}$ at x.

LEMMA 1 (ARONSZAJN-MILGRAM [3]). *Let A be an elliptic operator of order $2r$, and let $\{B_j\}_{j=1}^r$ be a normal set of boundary operators of orders $m_j < 2r$, respectively. Then we can find another normal set $\{B_j^*\}_{j=1}^r$ such that*

$$(1) \qquad (u, A^*v) = (Au, v)$$

for all v satisfying

$$(2) \qquad B_j^* v = 0 \quad \text{on } \dot{G}, \qquad j = 1, 2, \cdots, r,$$

if and only if

$$(3) \qquad B_j u = 0 \quad \text{on } \dot{G}, \qquad j = 1, 2, \cdots, r.$$

We call the set $\{B_j^*\}_{j=1}^r$ adjoint to $\{B_j\}_{j=1}^r$ relative to A. We also have

LEMMA 2. *The normal set $\{B_j\}_{j=1}^r$ covers A if and only if every normal set adjoint to $\{B_j\}_{j=1}^r$ relative to A also covers A.*

The boundary value problem $\pi(A, f, u_0, B_j)$ is to find a function $u \in C^\infty(\overline{G})$ such that $Au = f$ in G and $B_j u = B_j u_0$ on $\dot{G}, j = 1, 2, \cdots, r$. We now state our main result.

THEOREM. *Let A be a properly elliptic operator of order $2r$ which is covered by a normal set $\{B_j\}_{j=1}^r$. Let $\{B_j^*\}_{j=1}^r$ be any normal set adjoint to $\{B_j\}_{j=1}^r$ relative to A. Then the boundary value problem $\pi(A, f, u_0, B_j)$ has a solution for every f and u_0 if and only if the solution of $\pi(A^*, 0, 0, B_j^*)$ is unique.*

Outline of Proof. Complete $C^\infty(\overline{G})$ with respect to the norm $\| \ \|_{2r}$ and call the resulting Hilbert space H. For convenience we assume $u_0=0$. Set

$$[u, v] = (A^*u, A^*v) + \sum_{j=1}^{r} \langle B_j^*u, B_j^*v\rangle_{2r-m_j},$$

where the $\langle \ , \ \rangle_s$ are appropriate boundary inner products (cf. [1; 2; 8]). By Lemma 2 it follows that the set $\{B_j^*\}_{j=1}^r$ covers A (and consequently A^*). The results of [1; 2; 8] and the uniqueness of $\pi(A^*, 0, 0, B_j^*)$ imply

$$c^{-1}\|v\|_{2r}^2 \leqq [v, v] \leqq c\|v\|_{2r}^2 \qquad \text{for all } v \in H.$$

It then follows from the Lax-Milgram lemma [6] that there is a $g\in H$ such that

$$[g, v] = (f, v) \qquad \text{for all } v \in H.$$

Applying regularity theory similar to that of Nirenberg [7] and Browder [4] we then prove that $g\in C^\infty(\overline{G})$. Hence $u=A^*g$ satisfies $Au=f$ in G and $(u, A^*v)=(Au, v)$ for all $v\in C^\infty(\overline{G})$ satisfying (2). Thus u satisfies (3) by Lemma 1 and the proof is complete.

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