HOMOMORPHISMS AND IDEMPOTENTS OF GROUP ALGEBRAS

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Let $G$ be a locally compact abelian group. We denote by $M(G)$ the algebra of all finite complex-valued Borel measures on $G$. The algebra is normed by assigning to each measure its total variation, and the product or convolution of the measures $\mu$ and $\nu$ is defined by

$$(\mu \ast \nu)(E) = \int \int_{x+y \in E} d\mu(x) d\nu(y).$$

If a particular Haar measure is chosen on $G$, the subalgebra of absolutely continuous measures may be identified with $L(G)$, the algebra of absolutely integrable functions. The Fourier transform of a measure $\mu$ is a function $\hat{\mu}$ defined on $\hat{G}$, the dual group of $G$, by the formula

$$\hat{\mu}(\chi) = \int_{\hat{G}} \langle \chi, g \rangle d\mu(g),$$

where $\langle \chi, g \rangle$ denotes $\chi$ evaluated at $g$. Each $\chi$ thus yields a homomorphism of $M(G)$ onto the complex numbers. Every such homomorphism of $L(G)$ is obtained in this way.

Let $\varphi$ be a homomorphism of $L(G)$ into $M(H)$. After composing with $\varphi^*$, every homomorphism of $M(H)$ onto the complex numbers either is identically zero, or can be identified with a member of $\hat{G}$. We thus have a map $\Phi_\varphi$ from $\hat{G}$ into $\{\hat{G}, 0\}$, the union of $\hat{G}$ and the symbol 0, the latter to be considered as the point at infinity. Our main result is:

**Theorem 1.** For every homomorphism $\varphi$ of $L(G)$ into $M(H)$, there exist a finite number of cosets of open subgroups of $\hat{H}$, which we denote by $K_i$, and continuous maps $\psi_i : K_i \to \hat{G}$, such that

$$\psi_i(x + y - z) = \psi_i(x) + \psi_i(y) - \psi_i(z),$$

with the following property: there is a decomposition of $\hat{H}$ into the disjoint union of sets $S_j$, each lying in the Boolean ring generated by the sets $K_i$, such that on each $S_j$, $\phi_\star$ is either identically zero or agrees with some $\psi_i$, where $S_j \subseteq K_i$.

Conversely, for any such map of $\hat{H}$ into $\{\hat{G}, 0\}$, there is a homo-
morphism of $L(G)$ into $M(H)$ which induces it. The map carries $L(G)$ into $L(H)$ if and only if $\phi_1$ of every compact subset of $G$ is compact.

The main tool in the proof of Theorem 1 is the following lemma:

**Lemma.** If $G$ and $H$ are compact, then the graph of $\phi_*$, namely all pairs $(\phi_*(h), h)$ where $\phi_*(h)$ is not zero, is such that its characteristic function is the Fourier transform of a measure on $G \times H$.

The measure in the lemma must of course be an idempotent, that is, satisfy the equation $\mu * \mu = \mu$. The essential difficulty rests in the determination of all idempotent measures on a group.

**Theorem 2.** If $\mu$ is an idempotent measure, then $\mu$ is the characteristic function of a subset $E$ of $G$ which lies in the Boolean ring generated by cosets of open subgroups of $G$.

It is not difficult to deduce Theorem 1 from the above statements in the case in which $G$ and $H$ are compact. In the general case one shows that there is a natural extension of $\phi$ to a homomorphism of $L(G)$ into $M(H)$ where $G$ and $H$ are the Bohr compactifications of $G$ and $H$ respectively. It can then be shown that if $\hat{G}$ and $\hat{H}$ are taken in the discrete topology, Theorem 1 holds. However we know that $\phi_*$ is continuous and after some manipulation we can show that Theorem 1 holds in the original form.

Both Theorems 1 and 2 were known in special cases before. We note that Theorem 2 implies that the support of an idempotent measure is contained in a compact subgroup. Conversely, it is simple to reduce Theorem 2 to the case where $G$ is compact. If $\mu$ is absolutely continuous then it clearly is a finite sum of characters multiplied by Haar measure. The difficulty in general lies in analyzing the singular part of $\mu$. Here the main point is to show that $\mu$ has mass on a closed subgroup of infinite index. In the case that $\hat{G}$ has no elements of finite order, this statement is equivalent to saying that the set $E$ intersects some cyclic subgroup of $\hat{G}$ in an infinite set. For arbitrary $\hat{G}$ it is proved by more complicated means. In either case one needs a technique which will yield some restriction on the nature of the set $E$. It is of course true that $E$ can be an arbitrary finite set. Hence we can only hope to derive statements about the set $E$ which allow for a finite number of exceptions. Nevertheless, our technique yields statements concerning finite sums of characters. These we state for the circle group.

**Theorem 3.** For some $K$, whenever $c_j$ are such that $|c_j| \geq 1$, and $n_j$ are arbitrary distinct integers, we have
\[ \int_0^{2\pi} \left| \sum_{j=1}^N c_j e^{in_j^2} \right| dx > K \left( \frac{\log N}{\log \log N} \right)^{1/8}. \]

It is a conjecture of Littlewood that the inequality holds with \( K \log N \) on the right side. Previously, however, it was not even shown that the left side tended to infinity as a function of \( N \). Indeed in the course of the proof of Theorem 2 we actually need this fact. The proof of Theorem 3 is completely independent of any abstract considerations. It is accomplished by exhibiting finite linear combinations of exponentials, \( \phi_k \), such that \( |\phi_k| \leq 1 \) and yet, if \( \mu \) denotes the measure

\[ \sum c_j e^{in_j^2} dx, \]

\( \int \phi_k d\mu \) is large. We use some general lemmas concerning measures together with a combinatorial argument concerning the distribution of the integers \( n_j \). In the case of idempotent measures, the same type of argument is used to show that the set \( E \) has many finite sets \( P \) such that for all \( x \) in \( E \), there is some \( p \) in \( P \) such that \( x + p \) lies in \( E \). This, however, does not suffice to characterize \( E \) and further arguments are necessary. The details are too complicated to give here but will appear in forthcoming publications.

**References**