HOMOMORPHISMS AND IDEMPOTENTS OF GROUP ALGEBRAS

BY PAUL J. COHEN

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Let G be a locally compact abelian group. We denote by $M(G)$ the algebra of all finite complex-valued Borel measures on G. The algebra is normed by assigning to each measure its total variation, and the product or convolution of the measures μ and ν is defined by

$$(\mu * \nu)(E) = \iint_{x+y\in E} d\mu(x)d\nu(y).$$

If a particular Haar measure is chosen on G, the subalgebra of absolutely continuous measures may be identified with $L(G)$, the algebra of absolutely integrable functions. The Fourier transform of a measure μ is a function $\hat{\mu}$ defined on $\hat{G}$, the dual group of G, by the formula

$$\hat{\mu}(\chi) = \int_G (\chi, g)d\mu(g),$$

where (χ, g) denotes χ evaluated at g. Each χ thus yields a homomorphism of $M(G)$ onto the complex numbers. Every such homomorphism of $L(G)$ is obtained in this way.

Let ϕ be a homomorphism of $L(G)$ into $M(H)$. After composing with ϕ, every homomorphism of $M(H)$ onto the complex numbers either is identically zero, or can be identified with a member of $\hat{G}$. We thus have a map ϕ_* from $\hat{H}$ into $\{\hat{G}, 0\}$, the union of $\hat{G}$ and the symbol 0, the latter to be considered as the point at infinity. Our main result is:

THEOREM 1. *For every homomorphism ϕ of $L(G)$ into $M(H)$, there exist a finite number of cosets of open subgroups of $\hat{H}$, which we denote by K_i, and continuous maps ψ_i: $K_i \rightarrow \hat{G}$, such that*

$$\psi_i(x + y - z) = \psi_i(x) + \psi_i(y) - \psi_i(z),$$

with the following property: there is a decomposition of $\hat{H}$ into the disjoint union of sets S_j, each lying in the Boolean ring generated by the sets K_i, such that on each S_j, ϕ_* is either identically zero or agrees with some ψ_i, where $S_j \subset K_i$.

Conversely, for any such map of $\hat{H}$ into $\{\hat{G}, 0\}$, there is a homo-

morphism of $L(G)$ into $M(H)$ which induces it. The map carries $L(G)$ into $L(H)$ if and only if ϕ_*^{-1} of every compact subset of $\hat{G}$ is compact.

The main tool in the proof of Theorem 1 is the following lemma:

LEMMA. *If G and H are compact, then the graph of ϕ_*, namely all pairs $(\phi_*(h), h)$ where $\phi_*(h)$ is not zero, is such that its characteristic function is the Fourier transform of a measure on $G \times H$.*

The measure in the lemma must of course be an idempotent, that is, satisfy the equation $\mu * \mu = \mu$. The essential difficulty rests in the determination of all idempotent measures on a group.

THEOREM 2. *If μ is an idempotent measure, then $\hat{\mu}$ is the characteristic function of a subset E of $\hat{G}$ which lies in the Boolean ring generated by cosets of open subgroups of $\hat{G}$.*

It is not difficult to deduce Theorem 1 from the above statements in the case in which G and H are compact. In the general case one shows that there is a natural extension of ϕ to a homomorphism of $L(\overline{G})$ into $M(\overline{H})$ where $\overline{G}$ and $\overline{H}$ are the Bohr compactifications of G and H respectively. It can then be shown that if $\hat{G}$ and $\hat{H}$ are taken in the discrete topology, Theorem 1 holds. However we know that ϕ_* is continuous and after some manipulation we can show that Theorem 1 holds in the original form.

Both Theorems 1 and 2 were known in special cases before. We note that Theorem 2 implies that the support of an idempotent measure is contained in a compact subgroup. Conversely, it is simple to reduce Theorem 2 to the case where G is compact. If μ is absolutely continuous then it clearly is a finite sum of characters multiplied by Haar measure. The difficulty in general lies in analyzing the singular part of μ. Here the main point is to show that μ has mass on a closed subgroup of infinite index. In the case that $\hat{G}$ has no elements of finite order, this statement is equivalent to saying that the set E intersects some cyclic subgroup of $\hat{G}$ in an infinite set. For arbitrary $\hat{G}$ it is proved by more complicated means. In either case one needs a technique which will yield some restriction on the nature of the set E. It is of course true that E can be an arbitrary finite set. Hence we can only hope to derive statements about the set E which allow for a finite number of exceptions. Nevertheless, our technique yields statements concerning finite sums of characters. These we state for the circle group.

THEOREM 3. *For some K, whenever c_j are such that $|c_j| \geqq 1$, and n_j are arbitrary distinct integers, we have*

$$\int_0^{2\pi} \left| \sum_{j=1}^{N} c_j e^{i n_j x} \right| dx > K \left(\frac{\log N}{\log \log N} \right)^{1/8} .$$

It is a conjecture of Littlewood that the inequality holds with $K \log N$ on the right side. Previously, however, it was not even shown that the left side tended to infinity as a function of N. Indeed in the course of the proof of Theorem 2 we actually need this fact. The proof of Theorem 3 is completely independent of any abstract considerations. It is accomplished by exhibiting finite linear combinations of exponentials, ϕ_k, such that $|\phi_k| \leqq 1$ and yet, if μ denotes the measure

$$\sum c_j e^{i n_j x} dx,$$

$\int \phi_k d\mu$ is large. We use some general lemmas concerning measures together with a combinatorial argument concerning the distribution of the integers n_j. In the case of idempotent measures, the same type of argument is used to show that the set E has many finite sets P such that for all x in E, there is some p in P such that $x+p$ lies in E. This, however, does not suffice to characterize E and further arguments are necessary. The details are too complicated to give here but will appear in forthcoming publications.

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Massachusetts Institute of Technology