Buck then treat Sister Celine's polynomials including Bateman's Z_n and others as special cases. The authors also treat what they term Bessel polynomials, which are generalizations of one of Rainville's generalizations of what Krall and Frink called generalized Bessel polynomials. A few other polynomial sets are also considered in this chapter.

Chapter III (about 19 pages) is a study of the representation of functions regular at the origin. Again the Boas and Buck generalized Appell polynomials play a major role. Applications are made to Jacobi, Gegenbauer, Humbert, Lerch, Faber, and Sheffer polynomials among others.

Chapter IV (about 5 pages) gives applications to uniqueness theorems and certain functional equations. The book closes with an extensive bibliography and an index.

This book should be of interest to students of function theory. It certainly belongs in the library of everyone working with special functions.

EARL D. RAINVILLE

Eigenfunction expansions associated with second order differential equations, Part 2. E. C. Titchmarsh. Oxford University Press, 1958. 11+404 pp. $11.20.

One of the major difficulties facing an author of a book on eigenfunction expansions associated with partial differential equations is the decision as to which topics to discuss and at what level of generality to place the discussion. How general are the differential equations and domains to be considered? Should "expansion" mean "expansion in L^2" or something less or more general? Are the methods employed to be those of strictly classical analysis or those of abstract operator theory or a mixture of the two?

Professor Titchmarsh had faced similar, but perhaps simpler, versions of these questions when in 1946 he wrote Part 1 dealing with the ordinary differential equation $\{d^2/dx^2+\lambda-q(x)\}\psi(x)=0$. In the present volume, he stands by the decisions he made then and his object here is to treat the partial differential equation $\{\Delta+\lambda-q(x)\}\psi(x)=0$ on the entire x-space in a similar fashion, that is, by methods of classical analysis. The decision to deal only with the fundamental case of constant coefficients is probably a wise one. The complete avoidance of standard results and even the language of operator theory in Hilbert space leads a reader (particularly, an inexperienced reader, who may know the elements of operator theory) to lose a great deal of perspective.

The advantage of the author's treatment is, of course, that it permits a detailed, self-contained exposition. To further enhance the self-containment of the book, the author has added a final chapter containing discussions of a wide variety of miscellaneous topics such as the elements of potential theory, selection (Arzela, Helly) theorems, boundary values of analytic functions, Tauberian theorems, etc. The separation of these discussions from the main body of the book adds greatly to the readability of the book. The author uses another device to lessen the impact of some of the long calculations, namely, to give details only for the plane case and to indicate when necessary the modifications to obtain the results in the general case.

One feature of the exposition which causes slight difficulty is the fact that the conditions on $q(x)$ change from theorem to theorem and sometimes it is difficult to keep track of these changes. Also, the author never comments on whether the various conditions on q are essential or are imposed to facilitate the calculations at hand.

Most of the book is related to the following situation which is developed in the first few chapters: Let Δ be the Euclidean Laplacian and $q(x)$ a function of class C^1 in the entire x-space. Let $\lambda=\sigma+i\tau$, where $\tau\neq 0$. Then there exists a (not necessarily unique) Green's function $G(x, \xi, \lambda)$ such that the integral operator $f\rightarrow\phi$.

$$\phi(x) = \int G(x, \xi, \lambda)f(\xi)d\xi \tag{1}$$

is bounded on L^2; also, if f is of class C^1 on some open set, then ϕ is of class C^2 on that set and $\{\Delta+\lambda-q(x)\}\phi(x)=f(x)$. For real μ,

$$H(x, \xi, \mu) = \lim \int_0^\mu \operatorname{Im} G(x, \xi, \sigma + i\tau)d\sigma, \qquad \text{as } \tau \rightarrow 0, \tag{2}$$

exists for all $x, \xi(\neq x), \mu$ and as a function of μ is of bounded variation. The integral operator $f\rightarrow F$,

$$F(x) \equiv F(x, f, \mu) = \int H(x, \xi, \mu)f(\xi)d\xi \tag{3}$$

is bounded on L^2 and, in L^2,

$$F(x, f, \mu) - F(x, f, -\mu) \rightarrow \pi f(x) \text{ as } \mu \rightarrow \infty. \tag{4}$$

For f satisfying certain conditions including

$$\int_{|x|=R} (f\partial G/\partial r - G\partial f/\partial r)d\omega \rightarrow 0 \text{ as } R \rightarrow \infty,$$

one has the formula

$$(5)\quad F(x, f_1, \mu) = \int f(\xi)d\xi \int_0^\mu \sigma dH(x, \xi, \sigma) \text{ for } f_1 = -\{\Delta - q(x)\}f.$$

There is only a remark on the relation between spectral resolutions and (3), (4) and (5), and a verification of the usual orthogonality relations is given only under special conditions on q.

The formula (4) is called the "expansion theorem." One of Professor Titchmarsh's principal interests is the question of the validity of (4) at a fixed x in the sense of ordinary convergence or of a summation method (mentioned below) when the expansion formula reduces to a Fourier series.

The plan of this book follows closely that of Part I. (The first chapter in this volume is numbered XI, following I–X in Part 1.) In Chapter XI (Expansions in a rectangle), the Green's function $G=G_X$ and expansion for a rectangle $|x| \leqq X$, $|y| \leqq X$ with boundary conditions $\psi=0$ are obtained. G is first obtained for the case $q=0$ by Fourier series and then for arbitrary q by the method of successive approximations. The expansion theorem is obtained by residue calculus. In Chapter XII (Expansions in the whole plane), Green's functions are obtained by selection theorems applied to G_X as $X \to \infty$. Chapter XIII (Extension of the theory) involves modifications of the first two chapters (depending on iterations of the Green's functions) necessary to obtain results in n-space. In most of Chapter XIV (Variation of the eigenvalues), finite regions and the boundary condition $\psi=0$ are considered. The monotonous dependence of eigenvalues on $q(x)$ and on the region is discussed. These results are used to pass from expansions on a rectangle to expansions on certain types of general finite regions. In Chapter XV (Separable equations), formal formulae for the Green's functions in separable cases are justified under certain conditions on q. A result of Chapter XVI (The nature of the spectrum) is that $q(x) \to \infty$ as $|x| \to \infty$ implies that the spectrum consists of isolated eigenvalues of finite multiplicity clustering only at $\lambda = \infty$, so that the expansion formula reduces to a Fourier series. In Chapter XVII (The distribution of eigenvalues), $q(x)$ is subject to various conditions (including $q(x) \to \infty$ as $|x| \to \infty$) and asymptotic formulae for the number $N(\lambda)$ of eigenvalues not exceeding λ are discussed. Chapter XVIII (Convergence and summability theorems) is concerned with certain cases in which the expansion formula reduces to a Fourier series $f \sim \sum c_n \psi_n(x)$ and with conditions for convergence or summability (in the sense, $\lim \sum_{\lambda_n \leqq \lambda} (1 - \lambda_n/\lambda)^\alpha c_n \psi_n(x)$ as $\lambda \to \infty$) of this series. In Chapter XIX

(Perturbation theory), the coefficient function $q(x)$ is replaced by $q(x)+\epsilon s(x)$, where $q(x)\to\infty$ as $|x|\to\infty$, $s(x)\geqq 0$ and $\epsilon\geqq 0$. In Chapter XX (Perturbation theory involving continuous spectra), it is supposed that $q(x)\to\infty$, $s(x)\to-\infty$, $q(x)=o(|s(x)|)$ as $|x|\to\infty$. The main problem is an estimate of the contribution to the expansion of f, in the perturbed case, of the λ-values outside a small open set containing the discrete eigenvalues of the unperturbed problem. Chapter XXI (The case in which $q(x)$ is periodic) is devoted mainly to the 1-dimensional case. Finally, there is Chapter XXII (Miscellaneous theorems) mentioned above.

In a number of chapters, the general theory is illustrated by applications to interesting examples.

PHILIP HARTMAN

Linear operators. Part I: *General theory*. By Nelson Dunford and Jacob T. Schwartz, with the assistance of William G. Bade and Robert G. Bartle. New York, Interscience Publishers, 1958. 14+858 pp. $23.00.

In many ways this is an impressive book. The first way in which it is certain to make an impression on anyone who picks it up is by sheer size; an approximate word count reveals that it is only a little longer than *Doctor Zhivago*. Remember, however, that this is only the first volume, containing eight out of a total of twenty chapters. The work is intended to constitute an organic unit; the reasons for binding it in separate volumes are more practical than mathematical. Since at the time that this report is being written the second volume has not yet appeared, what follows refers to the first volume only.

The book makes use of several expository devices, which, while they are not new in concept, are here applied with such an astonishing degree of completeness and on such a gargantuan scale as to deserve special mention. On the end papers, for instance, there is a graph of the interdependence relations among the sections (of the first volume only) that is the most complicated thing of its kind the reviewer has ever seen. The graph is not embeddable into the plane, and it is not at all a trivial task to decipher the information it contains.

A helpful device is a black marginal arrow marking the theorems that may look insignificant but in fact play an important role in later developments.

At the end of most chapters there is a section of notes and remarks. These sections almost completely replace footnotes (a splendid idea). The remarks are not mere afterthoughts and references; they are