mination of the best (i.e., smallest) γ depends on the solution of the system $u_1=x$, $u_2=2^{-1/2}x+(u_1^2-1)^{1/2}$, $u_3=3^{-1/2}x+(u_2^2-1)^{1/2}$, $\cdots$. A detailed study of the asymptotic behavior of u_n, followed by numerical computations, showed that $\gamma=1.10649577\cdots$.

The author refers to Erdélyi, *Asymptotic expansions*, New York, Dover, 1956, for bibliography and more material on differential equations. Reference could also be made to the Russian monograph by Evgrafov (*Asymptotic estimates and entire functions*, Moscow, 1957), which contains many techniques and results for problems of a rather special kind.

R. P. BOAS, JR.

Multivalent functions. By W. K. Hayman. Cambridge Tract, no. 48. New York, Cambridge University Press, 1958. 8+151 pp. $4.00.

To the reviewer's knowledge there is no other topic of mathematics of comparable depth and sophistication to the theory of univalent functions in which it is possible to approach the same results by so many quite distinct methods (at least in the formal sense). To present a complete account of all aspects of the theory of univalent functions, to say nothing of its extensions to multivalent functions, would require an exceptionally large treatise. It is thus to be expected that any presentation of modest size will constitute a treatment of one or more particular aspects of the subject. In the present case the author quite naturally deals principally with that direction of the theory which is closest to his own work in the subject. This may be characterized as the study of univalent and multivalent functions by the traditional methods of analysis. This is not to say that the author confines himself to elementary methods (which serve to prove only the simplest results). Indeed he makes use in the proof of his principal objectives of the method of the extremal metric (at least in its primitive form of the length-area principle) and the method of symmetrization. However these methods are used to derive certain individual lemmas and theorems and the main stream of the argument follows the classical inequality proof pattern.

The present tract consists of six chapters. Of these the first and the last are somewhat apart from the main portion of the book. The first chapter is essentially a very brief survey of the most elementary parts of the theory of univalent functions. No further comment seems called for except to point out the one result here due to the author (Theorem 1.4) which plays an important role in later developments. Chapter six is an exposition of Löwner's parametric method and some of its applications. It depends on the intervening chapters only to the extent of one reference to Theorem 2.1. The presentation is essentially the standard one. However, the author has reworked it carefully, breaking it up into easy steps and reducing to explicit form some results usually taken from general principles assumed well known. The applications are the most familiar ones due to Löwner and Golusin. The author does not dwell at all on the least satisfactory aspect of Löwner's method, namely the matter of identifying and proving unique the extremal functions (see the translator's remark on p. 33 of the author's reference, Golusin [2]).

The other four chapters constitute the main part of the tract. Chapters two and three form a unit dealing with properties such as asymptotic growth and bounds for the coefficients of functions mean p-valent in the areal sense. These developments are due principally to Miss Cartwright, Biernacki and especially Spencer. The author has reworked and unified the technical details and added refinements and extensions. We give two typical results.

Suppose that $f(z)=\sum_0^\infty a_n z^n$ *is mean* p*-valent in* $|z|<1$. *Then*

$$M(r,f) < A(p)\mu_p(1-r)^{-2p} \qquad (0<r<1).$$

Also for $1\leqq n<\infty$

$$|a_n| < A(p)\mu_p n^{2p-1} \qquad (p>1/4),$$

$$|a_n| < A\,|a_0|\,n^{-1/2}\log(n+1) \qquad (p=1/4),$$

$$|a_n| < A(p)\,|a_0|\left[\frac{\log(n+1)}{n}\right]^{1/2} \qquad (0<p<1/4).$$

Here $M(r,f)=\max_{|z|=r}|f(z)|$, $\mu_p=\max_{\nu\leq p}|a_\nu|$, $A(p)$ depends only on p and A is an absolute constant.

In chapter four there is an account of the method of symmetrization (Steiner and circular symmetrization) on the general lines laid down by Pólya and Szegö. The author goes back to the very fundamentals and includes some details not to be found elsewhere, for example on Lipschitzian character of functions. Also the reviewer finds interesting the proof given of the decreasing of the capacity of a condenser under symmetrization. There follows a number of applications of symmetrization including results which play an essential role in the following chapter. These are largely the work of the author.

Chapter five deals with circumferentially mean p-valent functions. For these many results can be obtained in a sharp form which up to the present admit for areally mean p-valent functions only determination of the correct order of magnitude. These sharper results were in many cases first obtained by the author. When combined with results from chapters two and three they lead up to what is undoubtedly the most sophisticated result in the tract.

Suppose that $f(z)$ is circumferentially mean p-valent in $|z|<1$, that $p\lambda > 1/4$ and that $\phi(z) = [f(z)]^\lambda$ possesses a power series expansion

$$\phi(z) = z^\mu \sum_{n=-\infty}^{\infty} b_n z^n$$

in an annulus $1-2\delta<|z|<1$. Then

$$\lim_{n\to\infty} \frac{|b_n|}{n^{2p\lambda-1}} = \frac{\alpha^\lambda}{\Gamma(2p\lambda)}$$

where

$$\alpha = \lim_{r\to 1-} (1-r)^{2p} M(r,f).$$

Here $M(r,f)$ has the same meaning as previously.

The reviewer considers that the author has done an excellent job of presenting the portion of the subject which he has chosen to treat. While the Cambridge Tracts in general have a noticeable didactic bent the author has for the most part gone beyond the call of duty in providing background material and reducing to precise form certain auxiliary results often accompanied by some waves of the hand. Even more important he has organized and unified his account so that it is unusually easy reading for material containing so much technical detail. Despite this he has been able to reach close to the limit of knowledge in his particular direction of interest.

The book seems comparatively free of typographical errors although a number were observed. Against the background of the general high level of exposition occasional portions seem vague or not fully justified. We will not mention them all but only give several examples. At the top of page 40 the various implications are not valid for constants C_2, C_3, etc. without some further restrictions which come back ultimately to certain limitations on $A(p)$ which are never made explicit although satisfied. To be sure all steps can be argued out successfully. On page 69 in defining circular symmetrization it is never stated when the origin shall belong to the symmetrized set. In the statement of Theorem 4.8 it is not made clear what is meant by a half-line passing *through* a_0. With the most reasonable interpretation that this allows a_0 to be itself the centre of circular symmetrization the footnote on page 82 referring to the reviewer's result must be amended to allow in that case also D^* to be obtained from D by rigid rotation. On page 122 although it is asserted we are "making $r\to 1$" before the third display this passage to the limit does not seem to occur until the fourth display. Also, on a slightly different level, on page 122 in the asymptotic relation (6.6) the author has replaced the term $\log \beta_n$ by $-(1-\beta_n)$, yet when he applies the result on page 130 he has just to reverse this process. Finally on page 139 there is no discussion of the determination of $\arg f(z)/z$ in the inequality (6.21) and a similar condition obtains in several places afterwards.

As far as the choice of contents is concerned, in view of the obvious limitations, the reviewer regrets only that the author was not able to include some of his more recent results on asymptotic bounds for the coefficients. Also in chapter four he tacitly restricts the discussion of circular symmetrization to domains not containing the point at infinity which excludes the possibility of some applications to meromorphic functions.

The author makes no pretensions of biographical completeness. However there are several references he might well have added. One is to Faber's paper of 1920 in which a symmetrization method was first applied to the study of univalent functions. Another is to R. M. Robinson's paper of 1942 in which Löwner's method was extensively exploited.

JAMES A. JENKINS

An introduction to combinatorial analysis. By John Riordan. New York, Wiley, 1958. 10+244 pp. $8.50.

This is the first book on combinatorial analysis in forty-three years; the last was MacMahon's two-volume treatise in 1915–1916. It is, therefore, a very welcome arrival on the mathematical scene. Much of the material appears in book form for the first time. The emphasis throughout is on the methods of finding the number of ways in which a certain operation can be performed. Unsolved combinatorial problems are in abundance. For example it is not even known for $n=8$ how many distinct latin squares of order n there are, i.e., the number of distinct square matrices of order n containing the numbers from 1 to n in each row and column. Some mathematicians feel that combinatorial analysis is not a branch of mathematics but rather a collection of clever but unrelated tricks. This book successfully refutes that viewpoint.

The subject is, without doubt, one of the hardest in which to write an effective exposition. The reason for this is the fact that so much of the material occurs in an isolated fashion in so many different applications both to pure and applied mathematics and to other