RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A NEW TOPOLOGY FOR VON NEUMANN ALGEBRAS

BY PAUL C. SHIELDS[1]

Communicated by Walter Rudin, January 14, 1959

1. **Introduction.** The theory of von Neumann (v.N.) algebras is rich with topological structure, with many algebraic theorems being proved through the interplay of the various topologies. The strong and strongest topologies (von Neumann [7]) have the properties:

(1) The maps $a \to ab$ and $a \to ba$ are continuous.

(2) The map $(a, b) \to ab$ is continuous for $\|a\| \leq 1$. The weak and ultra-weak topologies (von Neumann [7]; Dixmier [2]) have property (1) and the following property:

(3) The map $a \to a^*$ is continuous.

The failure of property (2) for the ultra-weak topology and the failure of property (3) for the strongest topology cause much of the difficulties in handling these topologies. In this paper a new topology is introduced which has the properties (1)–(3) and also yields the same continuous linear functionals as the ultra-weak (hence also the strongest) topology. Using this topology we obtain a simpler proof of the Kaplansky Density Theorem (Kaplansky [3]). We also obtain new proofs of a pair of theorems due to Dixmier [2].

Throughout this paper we use a result due to Sakai [6], which states that a B^*-algebra $\mathfrak{A}$ has a representation as a v.N. algebra iff as a Banach Space $\mathfrak{A}$ is the adjoint of a Banach space X. We also obtain a theorem which shows that X is determined up to equivalence by $\mathfrak{A}$.

2. **The $\mu(\mathfrak{A}, X)$ topology.** Let X be a Banach space, $\mathfrak{A}$ its adjoint space and suppose that $\mathfrak{A}$ is a B^*-algebra. We denote the weak topology on X by $\sigma(X, \mathfrak{A})$ and the weak $*$-topology on $\mathfrak{A}$ by $\sigma(\mathfrak{A}, X)$. Sakai [6] shows that properties (1) and (3) hold for the $\sigma(\mathfrak{A}, X)$ topology. We define the mappings $a \to R_a$ and $a \to L_a$ of $\mathfrak{A}$ into $B(X)$ by $R_a x(b) = x(ba)$ and $L_a x(b) = x(ab)$. The map $a \to R_a (a \to L_a)$ is an

[1] This paper was prepared while the author was a National Science Foundation Predoctoral Fellow at Yale University. The author wishes to thank Professor C. E. Rickart for his help in the preparation of this paper. Detailed proofs and further results will be published elsewhere.

algebraic isometric antiisomorphism of $\mathfrak{A}$ into $B(X)$. The topology $\mu(\mathfrak{A}, X)$ is defined by taking as a basis for the neighborhoods of zero, sets of the form:

$$V_0(x_i, \cdots, x_n) = \{a \mid \|R_a x_i\| \leqq 1 \text{ and } \|L_a x_i\| \leqq 1, \text{ for } i=1, 2, \cdots n\}.$$

This gives a locally convex topology on $\mathfrak{A}$ which contains the $\sigma(\mathfrak{A}, X)$ topology.

LEMMA 1. *Let x be a fixed element of X. Then the maps $a \to R_a x (a \to L_a x)$ from $\mathfrak{A}$ into X are continuous when $\mathfrak{A}$ is given the $\sigma(\mathfrak{A}, X)$ topology and X is given the $\sigma(X, \mathfrak{A})$ topology.*

COROLLARY 1. *The sets $R_\Sigma x = \{R_a x \mid \|a\| \leqq 1\}$ and $L_\Sigma x = \{L_a x \mid \|a\| \leqq 1\}$ are compact with respect to the $\sigma(X, \mathfrak{A})$ topology.*

COROLLARY 2. *A linear functional on $\mathfrak{A}$ is continuous with respect to $\sigma(\mathfrak{A}, X)$ iff it is continuous with respect to $\mu(\mathfrak{A}, X)$.*

PROOF. This follows directly from Corollary 2 and the Mackey-Arens theorem (Mackey [**4**]; Arens [**1**]).

We note that a net $\{a_\alpha\}$ converges to a in the $\mu(\mathfrak{A},X)$ topology iff both $R_{a_\alpha} \to R_a$ and $L_{a_\alpha} \to L_a$ in the strong operator topology on $B(X)$.

3. **Applications.** Let f be a normal positive linear functional on $\mathfrak{A}$ (see Dixmier [**2**]). Let g be a positive functional on which is continuous with respect to the $\sigma(\mathfrak{A}, X)$ topology. Let $a_0 \in \mathfrak{A}$, $0 \leqq a_0 \leqq 1$, $a_0 \neq 0$ such that $f(a_0) < g(a_0)$. One can now adapt the proof in Dixmier [**2**, p. 15] to show that $\exists b_0 \in \mathfrak{A}$, $0 \leqq b_0 \leqq a_0$, $b_0 \neq 0$ such that $|f(ab_0)|^2 \leqq f(1) \cdot g(b_0 a * ab_0)$ for all $a \in \mathfrak{A}$. Thus if $a_\alpha \to a$ in the $\mu(\mathfrak{A}, X)$ topology with $\|a_\alpha\| \leqq 1$ we conclude that $f(a_\alpha b_0) \to f(ab_0)$. It follows from this that f is $\mu(\mathfrak{A}, X)$ continuous. Combining this with a result of Sakai [**6**], we obtain a theorem of Dixmier [**2**]:

PROPOSITION 1. *If f is a positive linear functional on $\mathfrak{A}$ then f is $\sigma(\mathfrak{A}, X)$ continuous iff f is normal.*

One can alter slightly a proof due to Pukansky [**5**] to obtain:

PROPOSITION 2. *Any $\sigma(\mathfrak{A}, X)$ continuous linear functional on $\mathfrak{A}$ is a linear combination of normal positive linear functionals.*

The following theorem now follows from Proposition 2:

THEOREM 1. *If Y is a Banach space such that $\hat{Y}$ (the adjoint of Y) is isometrically isomorphic to $\mathfrak{A}$, then Y is isometrically isomorphic to X.*

A consideration of hermitian linear functionals shows us that (3) holds for $\mu(\mathfrak{A}, X)$ and the identity $ab - a_0 b_0 = a(b - b_0) + (a - a_0) b_0$ shows us that (2) holds for $\mu(\mathfrak{A}, X)$. It is clear that (1) also holds for

$\mu(\mathfrak{A}, X)$. The following lemma can now be easily obtained:

LEMMA 2. *If $h(\lambda)$ is a bounded continuous real-valued function of a real variable, then the mapping $a \rightarrow h(a)$ is $\mu(\mathfrak{A}, X)$ continuous for hermition elements $a \in \mathfrak{A}$.*

As a corollary we have:

THEOREM (Kaplansky [3]). *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *subalgebras of $\mathfrak{A}$ such that $\mathfrak{A}_1$ is $\mu(\mathfrak{A}, X)$ dense in $\mathfrak{A}_2$ then the unit sphere of $\mathfrak{A}_1$ is $\mu(\mathfrak{A}, X)$ dense in the unit sphere of $\mathfrak{A}_2$.*

We note that if U is a $\mu(\mathfrak{A}, X)$ neighborhood of zero in $\mathfrak{A}$, then $1 \in U$ implies that $a \in U$ for $\|a\| \leqq 1$. Using this fact with a standard proof we obtain:

THEOREM 2. *Let $\mathfrak{B}$ be a *-subalgebra of $\mathfrak{A}$, containing 1, closed in the $\mu(\mathfrak{A}, X)$ topology. Let f_0 be a $\mu(\mathfrak{A}, X)$ continuous linear functional on $\mathfrak{A}$. Then there exists a $\mu(\mathfrak{A}, X)$ continuous linear functional f on $\mathfrak{A}$ such that $f(b) = f_0(b)$ for $b \in \mathfrak{B}$ and*

$$\sup_{a \in \mathfrak{A}; \|a\| = 1} |f(a)| = \sup_{b \in \mathfrak{B}; \|b\| = 1} |f_0(b)|.$$

As a corollary to this theorem we obtain a result which is essentially contained in a paper of Dixmier [2].

COROLLARY. *If $\mathfrak{A}$ and $\mathfrak{B}$ are as in Theorem 2 and f_0 is a positive normal linear functional on $\mathfrak{B}$ then there exists an extension to $\mathfrak{A}$ which is both positive and normal.*

Note added in proof: After this paper was submitted a proof of Theorem 2 was given in S. Sakai, *On linear functions of W*-algebras*, Proc. Jap. Acad. vol. 34 (1958) pp. 571–574.

REFERENCES

1. R. Arens, *Duality in linear spaces*, Duke Math. J. vol. 14 (1957) pp. 787–794.

2. J. Dixmier, *Formes linéaires sur un anneau d'operateurs*, Bull. Soc. Math. France vol. 81 (1953) pp. 9–39.

3. I. Kaplansky, *A theorem on rings of operators*, Pacific J. Math. vol. 1 (1951) pp. 227–232.

4. G. Mackey, *On convex topological linear spaces*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 519–537.

5. L. Pukansky, *The theorem of Radon-Nikodym in operator-rings*, Acta Sci. Math. Szeged vol. 15 (1954) pp. 149–156.

6. S. Sakai, *A characterization of W*-algebras*, Pacific J. Math. vol. 6 (1956) pp. 763–773.

7. J. von Neumann, *On a certain topology for rings of operators*, Ann. of Math. vol. 37 (1936) pp. 111–115.

YALE UNIVERSITY