RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ON THE INITIAL VALUE PROBLEM FOR PARABOLIC SYSTEMS OF DIFFERENTIAL EQUATIONS[1]

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1. **Introduction.** We consider the matrix linear differential operator

$$L \equiv P(x, y; D) - E \frac{\partial}{\partial y}$$

for $x=(x_1, \cdots, x_n) \in E^n$ and $y \in [y', y'']$, where E is the $N \times N$ identity matrix,

$$P(x, y; D) = \Big(\sum_{|k| \leq 2b} \overset{(k)}{A_{ij}} (x, y) D^{k} \Big) \; [i, j = 1, \cdots, N; b \geqq 1 \text{ an integer}],$$

$(k)=(k_1, \cdots, k_n)$ for non-negative integers k_j, $|k| = \sum_{j=1}^{n} k_j$, and $D^k = \partial^{|k|}/\partial x_1^{k_1} \cdots \partial x_n^{k_n}$. We will use D^m to denote an arbitrary D^k with $|k| = m$. Following Petrovskiĭ [6], we say that L is uniformly parabolic in $R = E^n \times [y', y'']$ if there exists a constant $\delta > 0$ such that all of the roots $\lambda = \lambda(x, y, \sigma)$ of

$$\det \Big\{ \Big(\sum_{|k|=2b} \overset{(k)}{A_{ij}} (x, y) (i\sigma)^{k} \Big) - \lambda E \Big\} = O \Big[(i\sigma)^{k} = \prod_{j=1}^{n} (i\sigma_j)^{k_j} \Big]$$

satisfy $\operatorname{Re} \lambda(x, y, \sigma) < -\delta$ for all $(x, y) \in R$ and real σ such that $\sum_{j=1}^{n} \sigma_j^2 = 1$. We assume throughout this paper that: (i) *L is uniformly parabolic in R* and (ii) *the coefficients $A_{ij}^{(k)}(x, y)$ of L are bounded uniformly continuous functions of y and satisfy a uniform Hölder condition* (*with exponent α, $0 < \alpha \leqq 1$*) *with respect to x in R*. Our main result is a uniqueness theorem for the solution of the initial value problem (i.v.p.)

$$(1.1) \qquad Lu = f(x, y) \text{ in } E^n \times (t, y'']; \qquad u(x, t) = g(x)$$

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for arbitrary $t \in [y', y'')$, where $f(x, y)$ and $g(x)$ are given N-vectors. In particular, we establish sufficient conditions for the unique representation of the solution of (1.1) in terms of the fundamental solution of $Lu=0$. The i.v.p. (1.1) has been investigated by various authors, notably Èĭdel'man [**3**; **4**] and Slobodetskiĭ [**8**] (see also Rosenbloom [**7**]).[2] These results will be described in the appropriate places below.

2. **Existence theory.** An $N \times N$ matrix $\Gamma(x, y; \xi, \eta)$ defined in $R \times R$ for $y > \eta$ is said to be a fundamental solution (f.s.) of $Lu=0$ if, as a function of x and y, Γ is a regular solution of $Lu=0$ for $y > \eta$ and if for suitable N-vectors $g(x)$

$$\lim_{y \to \eta+} \int \Gamma(x, y; \xi, \eta) g(\xi) d\xi = g(x).^{3}$$

Our results are based on the following

THEOREM 1. *If* (i) *and* (ii) *hold, then there exists a f.s.* $\Gamma(x, y; \xi, \eta)$ *of* $Lu=0$ *which can be written in the form*

$$\begin{aligned}\Gamma(x, y; \xi, \eta) &= G^{(\xi)}(x, y; \xi, \eta) + \int_\eta^y d\tau \int G^{(s)}(x, y; s, \tau)\Psi(s, \tau; \xi, \eta) ds \\ &\equiv G^{(\xi)}(x, y; \xi, \eta) + W(x, y; \xi, \eta),\end{aligned}$$

where

$$G^{(\zeta)}(x, y; \xi, \eta) = (2\pi)^{-n} \int e^{i(\sigma, x-\xi)} V^{(\zeta)}(y; \eta, \sigma) d\sigma \left[(\sigma, x) = \sum_{j=1}^{n} \sigma_j x_j\right],$$

the matrix $V^{(\zeta)}(y; \eta, z)$ *is the solution of* $(d/dy)V = P(\zeta, y; iz)V$; $V(\eta) = E[z = \sigma + i\gamma;\ \sigma, \gamma \in E^n]$, *and the matrix* $\Psi(x, y; \xi, \eta)$ *is the solution of the integral equation*

$$\Psi(x, y; \xi, \eta) = LG^{(\xi)}(x, y; \xi, \eta) + \int_\eta^y d\tau \int LG^{(s)}(x, y; s, \tau)\Psi(s, \tau; \xi, \eta) ds.$$

There exist constants $c_1 > c_2 > c_3 > c_4 > 0$ *and* $K_m > 0$ *depending only on* δ, $y'' - y'$ *and the bounds for the* $A_{ij}^{(k)}$ *such that*

[2] In the Russian literature (1.1) is called the Cauchy problem. Petrovskiĭ's definition of parabolicity is given for somewhat more general systems involving higher order derivatives with respect to y. However, the i.v.p. for such systems can be reduced to (1.1) by the introduction of additional dependent variables. In the papers cited above, Èĭdel'man deals with these more general systems.

[3] If the range of integration is not specified, the integral is understood to be taken over the whole E^n.

$$(2.1)\quad | D^mG^{(\zeta)}(x, y; \xi, \eta) | \leqq K_1(y-\eta)^{-(n+m)/2b}\exp(-c_1\rho(x, y; \xi, \eta)) \quad [m = 0, 1, \cdots, 2b+1],$$

$$(2.2)\quad | D^mG^{(\zeta)}(x', y; \xi, \eta) - D^mG^{(\zeta)}(x, y; \xi, \eta) | \leqq K_2(y-\eta)^{-(n+m+\mu)2/b} |x'-x|_q^\mu \exp(-c_2\rho(x, y; \xi, \eta)) \quad [m = 0, 1, \cdots, 2b; 0 \leqq \mu \leqq 1; | x' - x |_q \leqq (y-\eta)^{1/2b}],$$

$$(2.3)\quad | D^mW(x, y; \xi, \eta) | \leqq K_3(y-\eta)^{-(n+m-\alpha)/2b}\exp(-c_3\rho(x, y; \xi, \eta)) \quad [m = 0, 1, \cdots, 2b],$$

and

$$(2.4)\quad \begin{aligned} &| D^mW(x', y; \xi, \eta) - D^mW(x, y; \xi, \eta) | \\ &\leqq K_4(y-\eta)^{-(n+m-\alpha/2)/2b} | x' - x |_q^{\alpha/4} \\ &\cdot \{\exp(-c_4\rho(x', y; \xi, \eta)) + \exp(-c_4\rho(x, y; \xi, \eta))\} \quad [m = 0, 1, \cdots, 2b], \end{aligned}$$

where $q = 2b/(2b-1)$, $|x|_q = (\sum_{j=1}^n x_j^q)^{1/q}$, *and* $\rho(x, y; \xi, \eta) = (y-\eta)^{1-q} \cdot |x-\xi|_q^q$.

REMARK. The paramatrix $G^{(\zeta)}(x, y; \xi, \eta)$ is the f.s. of $P(\zeta, y; D)u - (\partial/\partial y)u = 0$ for any fixed $\zeta \in E^n$.

Theorem 1 (except for the Hölder continuity of D^mW) has been proved for the special case $N = b = 1$ in [**1**]; a slightly more general result for N, $b \geqq 1$ has been announced by Èĭdel'man [**4**]. In particular, Èĭdel'man uses a parametrix which depends only on the $A_{ij}^{(k)}$ for $|k| = 2b$ and consequently can dispense with the assumption that the continuity of the $A_{ij}^{(k)}$ with respect to y is uniform for $x \in E^n$ when $|k| < 2b$. Under this weaker hypothesis (2.4) is omitted and (2.2) holds with $G^{(\zeta)}$ replaced by Γ. Our hypothesis (ii) is essential for our uniqueness results and permits certain simplifications in the existence theory.

The proof of Theorem 1 for N, $b > 1$ is essentially the same as the proof in the case $N = b = 1$. The main difference is that in the latter case the parametrix is known explicitly, while in the former case its properties must be deduced from the corresponding properties of its Fourier transform (see, e.g., [**3**]). The principal difficulty in proving this theorem lies in proving the existence of $D^{2b}W$. We outline briefly the method of dealing with this point. Let $x_0 \in E^n$ be arbitrary and consider for all x which satisfy $|x - x_0|_q \leqq 1/2$

$$\int^* D^{2b}G^{(s)}(x, y; s, \tau)ds = \Big[-\int_{|s-x_0|_q=1}^{(n-1)} D^{2b-1}G^{(\zeta)}(x, y; s, \tau)d_\nu s$$
$$+ \int_{|s-x_0|_q\leqq 1} \{ D^{2b}G^{(s)} - D^{2b}G^{(\zeta)} \} ds + \int_{|s-x_0|_q\geqq 1} D^{2b}G^{(s)}ds \Big]_{\zeta=x}.^4$$

Since

$$| D^mG^{(\zeta')}(x, y; \xi, \eta) - D^mG^{(\zeta)}(x, y; \xi, \eta) |$$
$$\leqq K_5(y-\eta)^{-(n+m)/2b} | \zeta' - \zeta |_q^\alpha \exp(-c_1\rho(x, y; \xi, \eta))$$
$$[m = 0, 1, \cdots, 2b]$$

it follows that $|\int^* D^{2b}G^{(s)}ds| \leqq K_6(y-\tau)^{-1+\alpha/2b}$, where $K_6>0$ is independent of x_0, x, y, τ. Moreover, it can be shown that $|\Psi| \leqq K_7(y-\eta)^{-(n+2b-\alpha)/2b} \exp(-c_5\rho)$ and

$$| \Psi(x', y; \xi, \eta) - \Psi(x, y; \xi, \eta) | \leqq K_8(y-\eta)^{-(n+2b-\alpha/2)/2b} | x' - x |_q^{\alpha/2}$$
$$\cdot \{ \exp(-c_5\rho(x', y; \xi, \eta)) + \exp(-c_5\rho(x, y; \xi, \eta)) \},$$

where $c_1 > c_5 > 0$. From these facts we can show that $D^{2b}W$ exists and can be written in the form

$$(2.5)\qquad \begin{aligned} D^{2b}W(x, y; \xi, \eta) &= \int_\eta^{(y+\eta)/2} d\tau \int D^{2b}G^{(s)}(x, y; s, \tau)\Psi(s, \tau; \xi, \eta)ds \\ &+ \int_{(y+\eta)/2}^{y} d\tau \Big\{ \Big(\int^* D^{2b}G^{(s)}ds \Big) \Psi(x, \tau; \xi, \eta) \\ &+ \int D^{2b}G^{(s)}[\Psi(s, \tau; \xi, \eta) - \Psi(x, \tau; \xi, \eta)]ds \Big\} \end{aligned}$$

for $|x - x_0|_q \leqq 1/2$ and $y > \eta$. The estimate (2.3) follows immediately from (2.5). To prove (2.4) we use (2.5) together with

$$\Big| \int^* D^{2b}G^{(s)}(x', y; s, \tau)ds - \int^* D^{2b}G^{(s)}(x, y; s, \tau)ds \Big|$$
$$\leqq K_9(y-\tau)^{-1+\alpha/4b} |x' - x|_q^{\alpha/2}$$

for $|x' - x|_q \leqq 1/4$ and the observation that

[4] We use Hopf's notation [5] for the boundary integral over $|x_0 - s|_q = 1$.

$$\int D^m G^{(s)}(x+h, y; s, \tau)\, \Psi(s, \tau; \xi, \eta) ds$$

$$= \int D^m G^{(s+h)}(x, y; s, \tau)\Psi(s+h, \tau; \xi, \eta) ds.$$

Let $c, \epsilon > 0$ be chosen such that

$$| D^m \Gamma | \leqq K(y - \eta)^{-(n+m)/2b} \exp(-(c + \epsilon)\rho).$$

For any constant $\beta \geqq 0$ and $y' \leqq t \leqq y \leqq y''$ define

$$k(y, t) = c\beta/[c^{2b-1} - (y - t)\beta^{2b-1}]^{1/(2b-1)}.$$

If $D^i w(x, y)$ is a continuous function of $x \in E^n$ define

$$\mathfrak{N}_i[w; y, t] = \underset{E^n}{\text{l.u.b.}} \{ | \overset{i}{D} w(x, y) | \exp(-k(y, t) | x |_q^q)\}$$

and

$$\mathfrak{N}_{i+\mu}[w; y, t] = \underset{x', x \in E^n}{\text{l.u.b.}} \{\min[\exp(-k(y, t) | x' |_q^q), \exp(-k(y, t) | x |_q^q)]$$

$$\cdot | \overset{i}{D} w(x', y) - \overset{i}{D} w(x, y) | / | x' - x |_q^\mu\} \qquad [0 < \mu \leqq 1],$$

where the l.u.b. is taken with respect to all derivatives of order i. From Theorem 1 and the fact that $-c\rho(x, y; \xi, \eta) + k(\eta, t)|\xi|_q^q \leqq k(y, t)|x|_q^q$ for $y \in [t, y^*]$, where $y^* - t < (c/\beta)^{2b-1}$ and $y^* \leqq y''$, we prove

THEOREM 2. *Assume that*

(a) $\mathfrak{N}(g) = \text{l.u.b.}_{E^n}\{ |g(x)| e^{-\beta|x|_q^q}\} < \infty$ *and* $g(x)$ *continuous in* E^n.

(b) $f(x, y)$ *continuous in* $E^n \times (t, y^*]$, $\mathfrak{N}_0[f; y, t] < \infty$ *for* $y \in (t, y^*]$, *and* $\int_t^y \mathfrak{N}_0[f; \tau, t] d\tau < \infty$.

(c) *In every compact subregion of* $E^n \times (t, y^*]$ $f(x, y)$ *is Hölder continuous with respect to* x.

If (i) *and* (ii) *hold, then*

$$(2.6) \quad u(x, y) = \int \Gamma(x, y; \xi, t) g(\xi) d\xi - \int_t^y d\eta \int \Gamma(x, y; \xi, \eta) f(\xi, \eta) d\xi$$

is a regular solution of the i.v.p. (1.1) *in* $E^n \times [t, y^*]$. *There exists a constant* $K(\epsilon) > 0$ *depending on* ϵ, δ, $y'' - y'$ *and the bounds for the* $A_{ij}^{(k)}$, *but not on* y *or* t, *such that*

$$(2.7) \quad \mathfrak{N}_m[u; y, t] \leqq K(\epsilon) \left\{ (y - t)^{-m/2b} \mathfrak{N}(g) + \int_t^y (y - \eta)^{-m/2b} \mathfrak{N}_0[f; \eta, t] d\eta \right\}$$

$$[m = 0, 1, \cdots, 2b - 1].$$

For arbitrary $x_0 \in E^n$ *and all* x *which satisfy* $|x-x_0|_q \leqq 1/2$

$$
\begin{aligned}
D^{2b}u(x, y) = \int & D^{2b}\Gamma(x, y; \xi, t)g(\xi)d\xi \\
& - \int_t^{(y+t)/2} d\eta \int D^{2b}G^{(\xi)}(x, y; \xi, \eta)f(\xi, \eta)d\xi \\
& - \int_t^{y} d\eta \int D^{2b}W(x, y; \xi, \eta)f(\xi, \eta)d\xi \qquad (2.8) \\
& - \int_{(y+t)/2}^{y} d\eta \left\{ \left(\int^{*} D^{2b}G^{(\xi)}d\xi \right) f(x, \eta) \right. \\
& \left. + \int D^{2b}G^{(\xi)}[f(\xi, \eta) - f(x, \eta)]d\xi \right\}.
\end{aligned}
$$

Moreover, $D^m u$ *is Hölder continuous with respect to* x *in every compact subregion of* $E^n \times (t, y^*]$ *for* $m = 0, 1, \cdots, 2b$.

If we replace (c) *by* (c′) $\mathfrak{N}_{0+\mu}[f; y, t] < \infty$ *for* $y \in (t, y^*]$, *then*

$$
\begin{aligned}
\mathfrak{N}_{2b}[u; y, t] \leqq K(\epsilon) \Big\{ & (y-t)^{-1} \left(\mathfrak{N}(g) + \int_t^y \mathfrak{N}_0[f; \eta, t]d\eta \right) + (y-t)^{\mu/2b} \\
& \cdot \sup_{y+t \leqq 2\eta \leqq 2y^*} \mathfrak{N}_{0+\mu}[f; \eta, t] + \int_t^y (y-\eta)^{-1+\alpha/2b} \mathfrak{N}_0[f; \eta, t]d\eta \Big\}.
\end{aligned}
$$

If we replace (a) *and* (c) *by*

(a″) $\mathfrak{N}(g) < \infty$ *and* $\mathfrak{N}_\mu(g) = \operatorname*{l.u.b.}_{x', x \in E^n} \{ \min[\exp(-\beta |x'|_q^q),$

$$\exp(-\beta |x|_q^q)] \, |g(x') - g(x)| / |x' - x|_q^\mu \} < \infty$$

and

(c″) $\mathfrak{N}_{0+\mu}[f; y, t] < \infty$ *for* $y \in (t, y^*]$ *and* $\int_t^y \mathfrak{N}_{0+\mu}[f; \eta, t]d\eta < \infty$

respectively, then

$$
\begin{aligned}
\mathfrak{N}_{2b}[u; y, t] \leqq K(\epsilon) \Big\{ & (y-t)^{-1+\nu/2b} [\mathfrak{N}(g) + \mathfrak{N}_\mu(g)] \\
& + \int_t^y (y-\eta)^{-1+\nu/2b} (\mathfrak{N}_0[f; \eta, t] + \mathfrak{N}_{0+\mu}[f; \eta, t]) d\eta \Big\},
\end{aligned}
$$

where $\nu = \min(\mu, \alpha)$.

A result similar to Theorem 2 is proved in [1] for $N = b = 1$.

3. **Uniqueness theory.** We now consider the question of under what conditions (2.6) is the unique solution of (1.1). It has been shown by Èĭdel'man [**3**; **4**] and Slobodetskiĭ [**8**] that

THEOREM 3. *If* (i) *holds and if the coefficients* $A_{ij}^{(k)}$ *of* L *are such that the f.s. of the adjoint system* $\tilde{L}u=0$ *exists, then the only regular solution of*

$$(3.1)\qquad Lu = 0 \text{ in } E^n \times (t, y^*]; \qquad u(x, t) = 0$$

which satisfies $\mathfrak{M}_0[u; y, t] < \infty$ *for* $y \in [t, y^*]$ *is* $u(x, y) \equiv 0$.

It is clear that under our hypothesis (ii) Theorem 3 is not, in general, valid. However, by further restricting the class of solutions under consideration we obtain the following extension of Theorem 3, which is the main result of this investigation.

THEOREM 4. *If* (i) *and* (ii) *hold and if* $u(x, y)$ *is a regular solution of* (3.1) *in* $E^n \times [t, y^*]$ *such that* $(\alpha) D^m u(x, y)$ *is Hölder continuous with respect to* x *in* $E^n \times (t, y^*]$ *and* $(\beta) \mathfrak{M}_m[u; y, t] < \infty$ *for* $y \in (t, y^*]$ *for* $m = 0, 1, \cdots, 2b$ *then* $u(x, y) \equiv 0$.

The proof of Theorem 4, which we sketch below, is based on certain a priori estimates for the $D^m u$. The method was suggested by the Douglis-Nierenberg derivation of the Schauder estimates for elliptic systems [**2**], although the basic ideas occur in [**5**].

We first prove

LEMMA 1. *If* (i) *and* (ii) *hold and if* u *is a regular solution of* (3.1) *in* $E^n \times [t, y^*]$ *which satisfies* (α), (β) *and* $(\gamma) \int_t^y \mathfrak{M}_m[u; \tau, t] d\tau < \infty$ *for* $m = 0, 1, \cdots, 2b$ *then* $u(x, y) \equiv 0$.

Let $\zeta \in E^n$ be arbitrary. Then u is a regular solution in $E^n \times [t, y^*]$ of the i.v.p.

$$(3.2)\qquad \begin{aligned} \Lambda^{(\zeta)} u &\equiv P(\zeta, y; D)u - \frac{\partial}{\partial y} u = \{P(\zeta, y; D) - P(x, y, D)\} u(x, y) \\ &\equiv F^{(\zeta)}(x, y) \text{ in } E^n \times (t, y^*]; \qquad u(x, t) = 0 \end{aligned}$$

where, in view of (ii) and the conditions on $D^m u$, $F^{(\zeta)}(x, y)$ satisfies (b) and (c) of Theorem 2 uniformly for $\zeta \in E^n$. For fixed $\zeta \in E^n$, $G^{(\zeta)}(x, y; \xi, \eta)$ is a f.s. of $\Lambda^{(\zeta)} u = 0$ as a function of x, y and a f.s. of the adjoint system as a function of ξ, η. Thus, by Theorems 2 and 3

$$(3.3)\qquad D^m u(x, y) = -\int_t^y d\eta \int D^m G^{(\zeta)}(x, y; \xi, \eta) F^{(\zeta)}(\xi, \eta) d\xi \qquad [m = 0, 1, \cdots 2b - 1]$$

and, in view of (2.8),

$$
\begin{aligned}
D^{2b}u(x, y) = &- \int_t^{(y+t)/2} d\eta \int D^{2b}G^{(\zeta)}(x, y; \xi, \eta)F^{(\zeta)}(\xi, \eta)d\xi \\
&- \int_{(y+t)/2}^{y} d\eta \left\{ \left(\int^{*} D^{2b}G^{(\zeta)}d\xi \right) F^{(\zeta)}(x, \eta) \right. \\
&\left. + \int D^{2b}G^{(\zeta)}[F^{(\zeta)}(\xi, \eta) - F^{(\zeta)}(x, \eta)]d\xi \right\}
\end{aligned} \tag{3.3'}
$$

for arbitrary ζ, $x_0 \in E^n$ and $|x - x_0|_q \leqq 1/2$. In particular, choose $\zeta = x_0 = x$. Then, by (ii) and (3.2), $F^{(x)}(x, \eta) \equiv 0$ and $|F^{(x)}(\xi, \eta)| \leqq \hat{K}|x-\xi|_q^{\alpha} \sum_{j=0}^{2b} |D^j u(\xi, \eta)|$ for $\eta > t$, where $\hat{K} > 0$ is independent of x, ξ, η. Since the constant in (2.1) is also independent of x, y, ξ, η, it follows by applying these estimates to (3.3) and (3.3′) that there exists a constant $Q_1 > 0$ depending only on δ, ϵ, $y'' - y'$ and the bounds for the $A_{ij}^{(k)}$ such that

$$
\mathfrak{N}_j[u; y, t] \leqq Q_1 \int_t^y (y - \eta)^{-(j-\alpha)/2b} \sum_{m=0}^{2b} \mathfrak{N}_m[u; \eta, t] d\eta \qquad [j = 0, 1, \cdots, 2b]. \tag{3.4}
$$

Define

$$
\mathfrak{M}[u; y] = \int_t^y \sum_{m=0}^{2b} \mathfrak{N}_m[u; \eta, t] d\eta,
$$

where, by (γ), $\mathfrak{M}[u; y] < \infty$ for $y \in [t, y^*]$. Then integrating both sides of (3.4) with respect to y and interchanging the order of integrations on the right hand side, it is easy to show that there exists a constant $Q > 0$ depending on Q_1 and α such that

$$
\mathfrak{M}[u; y] \leqq Q(y - t)^{\alpha/2b}\mathfrak{M}[u; y] \text{ for } y \in [t, y^*].
$$

The proof of Lemma 1 can now be completed by standard arguments.

An immediate consequence of Theorem 2 and Lemma 1 is the following

LEMMA 2. *If* (i) *and* (ii) *hold and if f and g satisfy* (a″), (b) *and* (c″), *then* (2.6) *is the only regular solution of* (1.1) *in* $E^n \times [t, y^*]$ *which satisfies* (α), (β) *and* (γ).

Suppose now that u satisfies only the hypothesis of Theorem 4. For arbitrary (x, y) in $E^n \times (t, y^*]$ consider the i.v.p.

$$
Lv = 0 \text{ in } E^n \times (\tau, y^*]; \qquad v(x, \tau) = u(x, \tau), \tag{3.5}
$$

where $t<\tau\leqq(y+t)/2$. By Lemma 2 and Theorem 2, since u satisfies (α), (β) and (γ) in $E^n\times[\tau, y^*]$, $u(x, y)\equiv v(x, y; \tau)$ in $E^n\times[\tau, y^*]$ for any $\tau\in(t, (y+t)/2]$, where $v(x, y; \tau)=\int\Gamma(x, y; \xi, \tau)u(\xi, \tau)d\xi$. In view of continuity of $u(x, y)$ in $E^n\times[t, y^*]$ and of $\Gamma(x, y; \xi, \tau)$ as a function of τ, uniformly for $\tau\leqq(y+t)/2<y$, it follows that $\lim_{\tau\to t+} v(x, y; \tau)=0$ in $E^n\times(t, y^*]$. Thus $u(x, y)=0$ in $E^n\times(t, y^*]$ and, by continuity, the theorem is proved.

Finally, in view of Theorems 2 and 4 we have

THEOREM 5. *If* (i) *and* (ii) *hold and if* f *and* g *satisfy* (a), (b) *and* (c′), *then* (2.6) *is the only regular solution of* (1.1) *in* $E^n\times[t, y^*]$ *which satisfies* (α) *and* (β).

REMARK. The results of §§2 and 3 can be easily generalized by replacing $\mathfrak{N}_m[u; y, t]$ by

$$\mathfrak{N}_{p,m}[u; y, t] = \left(\int \mid D^m u(\xi, y)\mid^p \exp(-pk(y, t)\mid \xi\mid_q^q)d\xi\right)^{1/p}$$

for any $1\leqq p\leqq\infty$ (cf. [4; 8]).

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