

*Normed linear spaces.* By M. M. Day. (Ergebnisse der Mathematik und ihrer Grenzgebiete, new series, No. 21.) Berlin, Göttingen, Heidelberg, Springer, 1958. 8+139 pp., DM 28.

Banach's inspiring book on linear operators appeared in 1932 and has been the bible of functional analysts ever since; not only did he provide an invaluable account of the contemporary state of the subject, then in rapid development, especially through the work of the Polish school under his own leadership, but he asked the right questions, questions that have stimulated much of the later work in the field and whose effect has not yet been exhausted.

There have been so many new contributions to the field since 1932 that it has been difficult to keep track of them. The present work is a survey of the present state of knowledge about normed linear spaces, and its plan follows the customary *Ergebnisse* pattern; succinct proofs of the main results are given and numerous subsidiary results are stated without proof, but with references to the original papers. The author does not claim complete coverage of the recent literature, but little of real importance has been omitted within the limits that the author has set himself.

We now take the chapters one by one, summarizing their contents and adding a few remarks here and there. Chapter I begins with the purely algebraic aspects of the theory of real and complex linear spaces; the Hahn-Banach extension theorem is proved for linear functionals dominated by a sublinear functional. Then come the fundamental properties of linear topological spaces and locally convex spaces, and the elementary duality theory of the latter. The last section is devoted to partially ordered linear spaces and to separation theorems for convex sets; it includes an interesting generalization of the Krein-Rutman theorem, expressed in terms of the linear space and order structures without any reference to topology.

The fundamental properties of normed and metric linear spaces appear in Chapter II, and particular attention is paid to the complex of results associated with the principle of uniform boundedness, the closed graph theorem and the open mapping theorem; the extension of these results to locally convex spaces is also discussed. It is doubtful whether the translation of "tonnelé" as "kegely" will receive wide acceptance. The last section compares the various weak topologies on normed spaces, including the bounded weak topology.

Chapter III, on completeness, compactness and reflexivity, begins by showing how a locally convex space can be completed by adjoining those linear functionals on the dual space that are continuous

in the “almost weak\*” topology; Pták’s analysis of the open mapping theorem is discussed in this context. Various kinds of compactness are discussed and compared, and proofs are given for Eberlein’s theorem and some of its generalizations. The elementary properties of completely continuous operators are given, but the classical Riesz-Schauder theory is omitted. The chapter ends with numerous variants of the conditions for a Banach space to be reflexive.

Chapter IV covers the various types of unconditional convergence, including the Orlicz-Pettis and Dvoretzky-Rogers theorems; it describes the present state of the basis problem, both for Schauder bases and for unconditional bases. There is also a brief outline of the theory of tensor products, concentrating mainly on normed spaces. The reviewer would have welcomed a fuller discussion of biorthogonal systems in this chapter; for instance, he was unable to find any mention of Banach’s statement (*Opérations linéaires*, pp. 237–238) that a complete biorthogonal system can be constructed for any separable Banach space.

Chapter V begins with the Krein-Milman theorem on extreme points of convex sets and some of its generalizations, and also gives some other fixed-point theorems; the second half is devoted to spaces of continuous functions, and includes the Stone-Čech compactification and a number of the known characterization theorems for such spaces.

Chapter VI deals with partially ordered normed spaces; here we find the properties of abstract (M)-spaces and abstract (L)-spaces, characterization and representation theorems for them, the Stone-Weierstrass theorem and various extension and projection theorems. Properties of special spaces include the Dunford-Pettis theorem on weakly completely continuous operators and the Riesz-Kakutani-Markov theorem describing the dual of the space of continuous functions on a compact Hausdorff space.

Chapter VII is mainly devoted to the geometrical properties of the unit sphere of a normed space; rotundity, smoothness and other related properties are defined and analyzed, and the numerous geometrical characterizations of inner-product spaces are discussed.

The brief Chapter VIII, under the title “Reader’s Guide,” gives references to the literature on the parts of the subject that have been deliberately omitted from the book; among the topics mentioned are Banach algebras, almost periodic functions, integration and ergodic theory.

The style of the book is inevitably too condensed for continuous

reading, but all those interested in normed spaces will find it an indispensable work of reference. There are some ways in which the author might have helped the reader even more. For instance, one frequently finds a host of implication relations between a number of rather technical conditions, and one wishes that the author had committed himself to an opinion about which of these conditions are important enough to deserve serious attention. Again, the author sometimes gives prominence to the most general form of a result, but passes rapidly over a simpler form that will do the job just as well nine times out of ten. Finally, one does tend to lose one's way among the multitudinous abbreviations for various conditions, in spite of the 3-page index of symbols at the end of the book; most readers will be wanting to look up some particular point, rather than work steadily through the book, and they will often find that they have to consult a host of definitions before they have disentangled the result they are looking for.

In spite of these minor criticisms, the book is a most welcome one, and is sure to be widely used, especially as a guide to the immense mass of literature that has accumulated in the last generation.

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