RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ON INDEPENDENT GROUP CHARACTERS

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The theorem proved in this note, when taken in conjunction with the theory of the Bohr compactification of a locally compact abelian group (for which see [1]), provides density theorems for group characters which generalize the classical Kronecker and Kronecker-Weyl approximation theorems. The theorems thus obtained are in several respects extensions of those of Bundgaard [2]. An account of them will appear elsewhere.

If G is a locally compact abelian group then a *character* of G will be taken here to mean a continuous homomorphism χ of G into the circle group T. If G is discrete then its character group $H=G^*$ is compact and carries a unique Haar measure μ such that $\mu(H)=1$. If $\mathfrak{B}$ is the class of Borel subsets of H then $(H, \mathfrak{B}, \mu)$ is a probability field in the sense of Kolmogorov [3], and, for each $g\in G$, the function $\chi\to\chi(g)$ on H into T is a character of H, and is *a fortiori* a random variable for $(H, \mathfrak{B}, \mu)$.

If $\varnothing\neq S\subseteq G$ then $[S]$ will denote the subgroup of G generated by S, except that, if $S=(g)$, $[S]$ will also be denoted by $[g]$. The symbols $\mathbf{P}$, $\prod$ are used respectively for the restricted and unrestricted direct products. Thus if $(G_\lambda)_{\lambda\in\Lambda}$ is a family of discrete abelian groups then $\mathbf{P}_{\lambda\in\Lambda} G_\lambda$ is discrete, $\prod_{\lambda\in\Lambda} G_\lambda^*$ is compact, and each is the character group of the other for their natural pairing (see [4, §37]).

THEOREM. *Let $S=(g_\lambda)_{\lambda\in\Lambda}$ be a nonempty family of elements of G, let $K_\lambda=\{\chi(g_\lambda)\mid\chi\in H\}$ and let $\phi_S\colon H\to\prod_{\lambda\in\Lambda}K_\lambda$ be the homomorphism*

$$\chi\to(\chi(g_\lambda))_{\lambda\in\Lambda}\equiv\phi_S(\chi).$$

Then the following statements are equivalent:

(i) $$[S]=\mathop{\mathbf{P}}_{\lambda\in\Lambda}[g_\lambda];$$

(ii) $$\phi_S(H)=\prod_{\lambda\in\Lambda}K_\lambda;$$

(iii) *the functions* $\chi \to \chi(g_\lambda)$, $\lambda \in \Lambda$, *constitute an independent family of random variables for the probability field* $(H, \mathcal{B}, \mu)$.

We prove the implications (i)$\to$(ii)$\to$(iii)$\to$(i).

If (i) is true then $H/[S]^\perp = [S]^* = \prod_{\lambda\in\Lambda} [g_\lambda]^*$, where $[S]^\perp = \{\chi \in H \mid \chi(g) = 1 \text{ for all } g \in [S]\}$. For each $\chi \in H$ we can therefore find a unique family $(\chi_\lambda)_{\lambda\in\Lambda}$ with $\chi_\lambda \in [g_\lambda]^*$, $\lambda \in \Lambda$, such that $\chi(s) = \prod_{\lambda\in\Lambda} \chi_\lambda(s_\lambda)$ for all $s = \prod_{\lambda\in\Lambda} s_\lambda \in [S]$, where $s_\lambda \in [g_\lambda]$ for $\lambda \in \Lambda$. Condition (ii) follows at once.

Suppose next that (ii) is true. The group $K = \prod_{\lambda\in\Lambda} K_\lambda$ is compact and therefore carries a Haar measure ν for which $\nu(K) = 1$. The map $\phi_S: H \to K$ is an epimorphism and therefore $\mu(\phi_S^{-1}(A)) = \nu(A)$ for each Borel set $A \subseteq K$. Now let $\Lambda_0 = (\lambda_1, \lambda_2, \cdots, \lambda_n) \subseteq \Lambda$, where $1 \leqq n < \infty$, and let A_r be a Borel subset of K_{λ_r}, $1 \leqq r \leqq n$, and for each $\lambda \in \Lambda$ let ν_λ be the Haar measure on K_λ, normalized so that $\nu_\lambda(K_\lambda) = 1$. Suppose also that $B_\lambda = A_r$ for $\lambda = \lambda_r$, $1 \leqq r \leqq n$, and that $B_\lambda = K_\lambda$ for $\lambda \notin \Lambda_0$. Then, if $E_r = \{\chi \in H \mid \chi(g_{\lambda_r}) \in A_r\}$ and $E = \bigcap_{r=1}^n E_r$, we have, since ν is the product measure on K obtained from $(\nu_\lambda)_{\lambda\in\Lambda}$,

$$\mu(E) = \mu\left(\phi_S^{-1}\left(\prod_{\lambda\in\Lambda} B_\lambda\right)\right) = \prod_{\lambda\in\Lambda} \nu_\lambda(B_\lambda)$$

$$= \prod_{r=1}^n \nu_{\lambda_r}(A_r) = \prod_{r=1}^n \mu(E_r),$$

so that (iii) is true.

Suppose finally that (i) is false. Then we can find $\Lambda_0 = (\lambda_1, \lambda_2, \cdots, \lambda_n) \subseteq \Lambda$, with $1 \leqq n < \infty$, and integers k_r, for $1 \leqq r \leqq n$, such that $\prod_{r=1}^n g_{\lambda_r}^{k_r} = 1$, with $g_{\lambda_r}^{k_r} \neq 1$ for $r = 1, 2, \cdots, n$. This means that the character $f(\neq 1)$ of K defined by $f(\omega) = \prod_{r=1}^n \omega_{\lambda_r}^{k_r}$, $\omega = (\omega_\lambda)_{\lambda\in\Lambda} \in K$, is identically 1 on $\phi_S(H)$. But we can find $\omega \in K$ such that $f(\omega) \neq 1$, and then, by continuity of f, open sets $A_r \subseteq K_{\lambda_r}$, $1 \leqq r \leqq n$, such that $f(\omega') \neq 1$ when $\omega' \in \prod_{\lambda\in\Lambda} B_\lambda$, the B_λ being defined as before. Evidently $\phi_S^{-1}(\prod_{\lambda\in\Lambda} B_\lambda) = \emptyset$ and hence (again with the same notation) $E = \emptyset$, $\mu(E) = 0$. On the other hand

$$\prod_{r=1}^n \mu(E_r) = \prod_{r=1}^n \nu_{\lambda_r}(A_r) \neq 0,$$

and thus (iii) is false. Therefore statement (iii) implies (i), and the proof is complete.

I am indebted to Professor S. Kakutani for drawing my attention to Pontrjagin's proof of Kronecker's theorem. The foregoing proof

that statement (i) implies (ii) is essentially a rearrangement of part of Pontrjagin's argument (for which see [**4**, §37]).

REFERENCES

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2. S. B. E. Bundgaard, *Über die Werteverteilung der Charaktere Abelscher Gruppen*, Mat.-Fys. Medd. Danske Vid. Selsk. vol. 14 (1936–1937) no. 4.

3. N. Kolmogorov, *Foundations of the theory of probability* (translation of the German original of 1933), New York, 1950.

4. L. S. Pontrjagin, *Topologische Gruppen* I and II (translation of the Russian second edition of 1954), Leipzig, 1957–1958.

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