ON SOLUTIONS OF RIEMANN’S FUNCTIONAL EQUATION

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1. Let \( \{\lambda_n\}, \{\mu_n\} \) \( (n \geq 1) \) be two given sequences of positive numbers increasing to infinity, and let \( \delta > 0 \). We call the triplet \( \{\delta, \lambda_n, \mu_n\} \) a label. If \( s \) is a complex variable, \( s = \sigma + i\tau \), we speak of a solution of Riemann’s functional equation

\[
\pi^{-s/2} \Gamma \left( \frac{1}{2} s \right) \phi(s) = \pi^{-(\delta-s)/2} \Gamma \left( \frac{1}{2} (\delta - s) \right) \psi(\delta - s),
\]

pertaining to the label \( \{\delta, \lambda_n, \mu_n\} \), if there exist two Dirichlet series \( \phi(s) = \sum a_n\lambda_n^{-s}, \psi(s) = \sum b_n\mu_n^{-s} \) (\( a_n \) and \( b_n \) complex) which do not vanish identically, and which admit finite abscissae of absolute convergence, and a function \( \chi(s) \) which is holomorphic and uniform in a domain \( |s| > R \), such that \( \lim_{|\tau| \to \infty} \chi(\sigma + i\tau) = 0 \) uniformly in every segment \( \sigma_1 \leq \sigma \leq \sigma_2 \), and such that, for some pair of real numbers \( \alpha, \beta \), we have

\[
\chi(s) = \begin{cases} 
\pi^{-s/2} \Gamma \left( \frac{1}{2} s \right) \phi(s), & \text{for } \sigma > \alpha, \\
\pi^{-(\delta-s)/2} \Gamma \left( \frac{1}{2} (\delta - s) \right) \psi(\delta - s), & \text{for } \sigma < \beta.
\end{cases}
\]

In three papers published recently, Bochner and Chandrasekharan [2], Chandrasekharan and Mandelbrojt [3], and Kahane and Mandelbrojt [4], have studied the problem of finding an upper bound for the number of linearly independent solutions of equation (1.1). Their results enable one to establish in certain cases a unique solution, and in certain others to deduce that the sequences \( \{\lambda_n\}, \{\mu_n\} \) are periodic. In this note, which is a sequel to [3], we shall consider certain simple conditions which would ensure that \( \delta = 1 \). Let

\[
D^* = \lim \sup (n/\mu_n), \quad h^* = \lim \inf (\mu_{n+1} - \mu_n).
\]

We prove the following results.

**Theorem 1.** If \( h^* \cdot h^* = 1, \delta \) is an odd integer, and equation (1.1) has a solution, then \( \lambda_{n+1} - \lambda_n = h^*, \text{ and } \mu_{n+1} - \mu_n = h^* \), for every \( n \geq 1 \). In particular, if \( h^* = h^* = 1, \delta \) is an odd integer, and equation (1.1) has a solution, then \( \lambda_{n+1} - \lambda_n = 1, \text{ and } \mu_{n+1} - \mu_n = 1 \) for every \( n \geq 1 \).
THEOREM 2. If \( h_n > 0 \), \( \delta \) is an odd integer, \( b_n = O(1) \), and equation (1.1) has a solution, then \( \delta = 1 \).

THEOREM 3. Let \( h_n > 0 \), and let \( \delta \) be an odd integer. If simultaneously, \((\sum a_n \lambda_n^s, \sum b_n \mu_n^s)\) is a solution of equation (1.1) with the label \((\delta, \lambda_n, \mu_n)\), and \((\sum c_n \lambda_n^s, \sum d_n \mu_n^s)\) is a solution with the label \((\delta, \lambda_n', \mu_n)\) for some \( (\lambda_n') \), and \((\sum e_n \lambda_n''^s, \sum f_n d_n h_n^{-s})\) is also a solution with the label \((\delta, \lambda_n'', \mu_n)\) for some \( (\lambda_n'') \), where \( (b_n/d_n) = o(\mu_n) \); then \( \delta = 1 \).

2. For the proof of these theorems we require a number of lemmas.

LEMMA 1. Equation (1.1) implies, for a sufficiently large integer \( r \), the following relation:

\[
\Gamma \left\{ \frac{1}{2} (\delta + 1) \right\} \pi^{-(\delta+1)/2} \sum_{n=1}^{\infty} a_n \left[ \frac{d^2r}{ds^2} \frac{s}{(s^2 + \lambda_n^2)^{\delta+1/2}} \right] - K_r(s) = (2\pi)^{2r} \sum_{n=1}^{\infty} b_n \lambda_n^{2r} \exp(-2\pi \mu_n s),
\]

for Re \( s > 0 \), where \( K_r(s) \) is holomorphic on the surface on which \( \log s \) is defined, and \( K_r(s) = O(|s|^{-\epsilon}) \), \( \epsilon > 0 \), as \( s \to \infty \) in any angle \( |\arg s| \leq \theta_0 \).

This has been proved by Bochner and Chandrasekharan [Theorem 2.1, p. 344]. By the definition of functional equation (1.1) it follows that the Dirichlet series on the right of (2.1) converges absolutely for \( \sigma > 0 \), and from (2.1) it follows that the singularities of its sum-function are situated symmetrically on the imaginary axis \( \sigma = 0 \), at the points \( (\pm \lambda_n) \), and also possibly at the origin, which we may, for convenience, designate as \( \lambda_0 \).

LEMMA 2. If \( D^\mu < \infty \), and equation (1.1) has a solution, then \( D^\lambda \cdot D^\mu \geq 1 \), and \( h_\lambda \cdot h_\mu \leq 1 \). (With the understanding that if \( D^\mu = 0 \), then \( D^\lambda = +\infty \).)

This is an immediate consequence of a theorem of Chandrasekharan and Mandelbrojt [3, Theorem 1, p. 289] which implies [loc. cit., p. 290, ll. 6–9] that if \( D^\mu < \infty \), and equation (1.1) is satisfied, then \( \lambda_{n+1} - \lambda_n \leq D^\mu \) for every \( n \geq 1 \), that is, \( \lambda_n \leq n \cdot D^\mu \), or \( n/\lambda_n \geq 1/D^\mu \), or \( D^\lambda \cdot D^\mu \geq 1 \). Since we have \( D^\mu \cdot h_\mu \leq 1 \), it follows that \( h_\lambda \cdot h_\mu \leq 1 \).

LEMMA 3. If \( h_\mu > 0 \), \( \delta \) is an odd integer, and equation (1.1) has a solution, then \( \delta = 1 \) or 3.

This is a result of Kahane and Mandelbrojt [4, Theorem 3, pp. 71–72].
Lemma 4. If \( h_\mu > 0 \), and \( \delta = 1 \) or \( 3 \), and equation (1.1) has a solution, then \( \mu_{n+1} - \mu_n \geq h_\mu \). And for \( \sigma < 0 \), the analytic continuation of the series

\[ \Psi(s) = \begin{cases} \sum b_n \exp(-2\pi \mu_n s) & \text{if } \delta = 1, \\ \sum b_{n\mu_n}^{-1} \exp(-2\pi \mu_n s) & \text{if } \delta = 3, \end{cases} \]

which is a uniform function, is given by the series \(- \sum_{n=1}^\infty b_n \exp(+2\pi \mu_n s)\), and the only singularities of \( \Psi(s) \) are simple poles at the points \( \pm i\lambda_n \), \( n = 0, 1, 2, \ldots \).

A result proved earlier by Chandrasekharan and Mandelbrojt [3, Theorem 3, p. 292] gives the Dirichlet series representation of \( \Psi(s) \) in the negative half-plane as \( \sum c_n \exp(2\pi \mu'_n s) \) but it is easy to see that \( c_n = -b_n \), and \( \mu_n = \mu'_n \), if one observes that by Agmon’s theorem, used in that proof, the origin is a simple pole for the residual function \( K_r(s) \) in (2.1). This fact is also obvious from the paper by Kahane and Mandelbrojt [4].

Lemma 5. If \( h_\mu > 0 \), and \( f(s) = \sum_0^\infty B_n \exp(-2\pi \mu_n s) \) has \( \sigma = 0 \) as its abscissa of absolute convergence, and the only singularities of \( f(s) \) on a segment of the imaginary axis of length greater than \( h_\mu^{-1} \) are poles of greatest order \( q \), then \( B_n = O(\mu_n^{-q}) \).

This is a tauberian theorem of S. Agmon [1, Theorem 4.3(C)].

Lemma 6. If \( D^\sigma < \infty \), and \( b_n = O(\mu_n^{q-1}) \), then for \( \sigma > 0 \), we have

\[ f(s) = \sum_{n=1}^\infty b_n \exp(-2\pi \mu_n s) = O(\sigma^{-q}). \]

If in the hypothesis we have \( b_n = o(\mu_n^{q-1}) \), then the conclusion is \( f(s) = o(\sigma^{-q}) \).

(i) Since \( D^\sigma < \infty \), we have \( \mu_n > Ln \) for every \( n \), where \( L \) is some constant. Now, for \( \sigma > 0 \), we have

\[ |f(s)| \leq C \cdot \sum_{n=1}^\infty \mu_n^{q-1} \exp(-2\pi \mu_n \sigma) \]

\[ \leq C \cdot (2\pi \sigma)^{1-q} \sum_{n=1}^\infty (2\pi \mu_n \sigma)^{q-1} \exp(-2\pi \mu_n \sigma). \]

The term \((2\pi \mu_n \sigma)^{q-1} \exp(-2\pi \mu_n \sigma)\) decreases (as \( \mu_n \) increases), when \( 2\pi \mu_n \sigma > q - 1 \). Let \( n_\sigma \) be the smallest \( n \) for which we have \( 2\pi L n \sigma > q - 1 \); in other words, for \( n = 1, \ldots, n_\sigma - 1 \), we have \( 2\pi L n \sigma \leq q - 1 \). Then
\[ \sum_{n} (2\pi \mu_n \sigma)^{q-1} \exp (-2\pi \mu_n \sigma) \leq \sum_{n} (L \cdot 2\pi n \sigma)^{q-1} \exp (-2\pi L n \sigma) \]

\[ = O \left( \sigma^{q-1} \sum_{n} n^{q-1} \exp (-2\pi L n \sigma) \right) \]

\[ = O(\sigma^{-1}) \]

while

\[ \sum_{1}^{n-1} (2\pi \mu_n \sigma)^{q-1} \exp (-2\pi \mu_n \sigma) \leq \max_{x \geq 0} [x^{q-1} e^{-x}] \cdot (n_\sigma - 1) \]

\[ \leq K \cdot (n_\sigma - 1) = O(\sigma^{-1}). \]

Hence \( f(s) = O(\sigma^{-q}) \).

(ii) In case \( b_n = o(\mu_n^{-1}) \), let \( n_\sigma \) be the smallest \( n \) such that \( (2\pi n \sigma) > (q-1)\sigma^{1/2} \). Then, as before,

\[ \left| \sum_{1}^{n-1} b_n \exp (-2\pi \mu_n s) \right| = O(\sigma^{1-q} (n_\sigma - 1)) = O(\sigma^{1/2-q}), \]

and, since \( n_\sigma \to \infty \), as \( \sigma \to 0 \), we have

\[ \left| \sum_{n} b_n \exp (-2\pi \mu_n s) \right| = o(\sigma^{1-q}) \cdot \left| \sum_{n} (2\pi \mu_n \sigma)^{q-1} \exp (-2\pi \mu_n \sigma) \right| \]

\[ = o(\sigma^{-q}). \]

Hence \( f(s) = o(\sigma^{-q}) \).

3. We shall now indicate the proofs of Theorem 1 to 3.

**Proof of Theorem 1.** We remark that by Lemma 2, we have

\[ h_\lambda \cdot h_\mu \leq 1. \]

If \( h_\lambda \cdot h_\mu = 1 \), then we have \( h_\lambda > 0 \), and \( h_\mu > 0 \), so that \( D^\lambda < \infty \), and \( D^\mu < \infty \). Hence, as in the proof of Lemma 2, we have \( \mu_{n+1} - \mu_n \leq D^\lambda \leq h_\lambda^{-1} = h_\mu \), and \( \lambda_{n+1} - \lambda_n \leq D^\mu \leq h_\mu^{-1} = h_\lambda \). Since \( \delta \) is odd, we have, by Lemma 3, \( \delta = 1 \) or \( 3 \). Now, by the first part of Lemma 4, we have \( \mu_{n+1} - \mu_n \leq h_\mu \), and \( \lambda_{n+1} - \lambda_n \leq h_\lambda \), which lead to the desired result.

**Proof of Theorem 2.** By Lemma 3, we have \( \delta = 1 \) or \( 3 \). We shall show that the case \( \delta = 3 \) is incompatible with the hypotheses. Consider the series \( f(s) = \sum b_n \mu_n^{2r} \exp (-2\pi \mu_n s) \) in Lemma 1. Since \( b_n = O(1) \), we have, by Lemma 6, \( f(s) = O(\sigma^{-2r-1}) \), for \( \sigma > 0 \). On the other hand, in a neighborhood of a pole, say \( s = \delta \lambda_n, \ n \geq 1 \), we have \( |f(s)| > c \cdot |\sigma|^{-p} \), where \( p \) is the order of the pole, hence an integer, with \( p = (1/2)(\delta+1)+2r \). For these two estimates to be compatible, we should have \( \delta = 1 \).

**Proof of Theorem 3.** It is sufficient to show that \( \delta = 3 \) is impos-
sible. If $\delta = 3$, then by Lemmas 5 and 1, we have $b_n = O(\mu_n)$, $d_n = O(\mu_n)$ and $b_n d_n = O(\mu_n)$. But by hypothesis, $|b_n| \leq \epsilon_n \cdot |d_n| \cdot \mu_n$, where $\epsilon_n > 0$, and $\epsilon_n \to 0$ as $n \to \infty$. That is, $b_n d_n \geq |b_n|^2 \cdot (\mu_n \epsilon_n)^{-1}$. We now observe that $b_n = o(\mu_n)$ is impossible, because otherwise, by Lemma 6(ii), we should have $f(s) = o(\sigma^{-2})$, which contradicts the fact that $|f(\lambda_n + \sigma)| > c \cdot \sigma^{-2}$ for $\sigma > 0$. Hence there exists a sequence $(\mu_n)$ such that

$$|b_n| > \epsilon_n^{1/3} \cdot \mu_n,$$

which, together with the inequality for $b_n d_n$ obtained above, yields $|b_n \cdot d_n| \geq \epsilon_n^{2/3} \cdot \mu_n^{2/3} \cdot (\mu_n \cdot \epsilon_n)^{-1} \geq \mu_n \cdot \epsilon_n^{-1/3}$. But this contradicts the fact that $b_n d_n = O(\mu_n)$.

References