A REMARK ON CURVATURE AND THE DIRICHLET PROBLEM

BY M. S. NARASIMHAN

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1. Introduction. Let $M$ be a compact oriented Riemannian manifold with positive definite Ricci curvature. By a well-known theorem of Bochner-Myers [2, §26, p. 132] there are no nonzero harmonic forms of degree one on $M$. This result implies that "Dirichlet's problem" in the sense of §6 and §4 of reference [1] is solvable in $M$ and that there exists the Green's form of degree one on $M$ which is an elementary kernel for the Laplacian $\Delta$ (on 1-forms) [2, §31]. We shall point out in this note how, in this form, the result can be generalized to noncompact manifolds. We use the notations of [2].

2. Theorem. Let $M$ be an oriented $C^\infty$ Riemannian manifold countable at infinity. We assume that the mean curvature is positive and bounded away from zero, that is we assume that there exists a constant $C>0$ such that

$$R(v, v) \geq Cg(v, v)$$

for every tangent vector $v$, $R(v, v)$ denoting the Ricci form and $g(v, v)$, the metric form. Then Dirichlet's problem for 1-forms is solvable on $M$ and there exists the Green's form of degree one.

Proof. Referring to Proposition IV and §5 of [1], we have only to prove the Poincaré inequality for $C^\infty$ 1-forms with compact supports. Let $\alpha = (\alpha_1, \cdots, \alpha_k, \cdots, \alpha_n)$ be a $C^\infty$ 1-form with compact support. Then

$$(\Delta \alpha)_k = -\nabla_i \nabla_i \alpha_k - R_k^h \alpha_k.$$ 

We have (see [2, p. 132]),

$$(\alpha, \Delta \alpha)_{L^2} = -\int \alpha_k \nabla_i \nabla_i \alpha_k \ast 1 - \int R_k^h \alpha_k \ast 1$$

$$= \int \nabla_i \alpha_k \nabla_i \alpha_k \ast 1 - \int R_k^h \alpha_k \ast 1,$$

using integration by parts. By assumption we have

$$-R_k^h \alpha_k \ast 1 \geq C g_k^h \alpha_k \ast 1$$

and we have

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$$\int \nabla \alpha^k \nabla i \alpha_k * 1 \geq 0.$$ Consequently

$$(\alpha, \Delta \alpha)_{L^2} \geq C \int g_{ab} \alpha^a \alpha^b * 1$$

that is,

$$(d \alpha, d \alpha)_{L^2} + (\partial \alpha, \partial \alpha)_{L^2} \geq C(\alpha, \alpha)_{L^2}$$

which is Poincaré’s inequality for 1-forms with compact supports

REFERENCES


Tata Institute of Fundamental Research, Bombay, and
Centre National de la Recherche Scientifique, Paris