A REMARK ON CURVATURE AND THE DIRICHLET PROBLEM

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1. **Introduction.** Let M be a compact oriented Riemannian manifold with positive definite Ricci curvature. By a well-known theorem of Bochner-Myers [2, §26, p. 132] there are no nonzero harmonic forms of degree one on M. This result implies that "Dirichlet's problem" in the sense of §6 and §4 of reference [1] is solvable in M and that there exists the Green's form of degree one on M which is an elementary kernel for the Laplacian Δ (on 1-forms) [2, §31]. We shall point out in this note how, in this form, the result can be generalized to noncompact manifolds. We use the notations of [2].

2. THEOREM. *Let M be an oriented C^∞ Riemannian manifold countable at infinity. We assume that the mean curvature is positive and bounded away from zero, that is we assume that there exists a constant $C>0$ such that*

$$R(v, v) \geqq Cg(v, v)$$

for every tangent vector v, $R(v, v)$ denoting the Ricci form and $g(v, v)$, the metric form. Then Dirichlet's problem for 1-forms is solvable on M and there exists the Green's form of degree one.

PROOF. Referring to Proposition IV and §5 of [1], we have only to prove the Poincaré inequality for C^∞ 1-forms with compact supports. Let $\alpha=(\alpha_1, \cdots, \alpha_k, \cdots, \alpha_n)$ be a C^∞ 1-form with compact support. Then

$$(\Delta\alpha)_k = -\nabla^i\nabla_i\alpha_k - R_k^h\alpha_k.$$

We have (see [2, p. 132]),

$$(\alpha, \Delta\alpha)_{L^2} = -\int \alpha^k\nabla_i\nabla^i\alpha_k * 1 - \int R_k^h\alpha_h\alpha^k * 1$$

$$= \int \nabla_i\alpha^k\nabla^i\alpha_k * 1 - \int R_{hk}\alpha^h\alpha^k * 1,$$

using integration by parts. By assumption we have

$$-R_{hk}\alpha^h\alpha^k \geqq Cg_{hk}\alpha^h\alpha^k$$

and we have

$$\int \nabla_i \alpha^k \nabla^i \alpha_k * 1 \geqq 0.$$

Consequently

$$(\alpha, \Delta\alpha)_{L^2} \geqq C \int g_{hk} \alpha^h \alpha^k * 1$$

that is,

$$(d\alpha, d\alpha)_{L^2} + (\partial\alpha, \partial\alpha)_{L^2} \geqq C(\alpha, \alpha)_{L^2}$$

which is Poincaré's inequality for 1-forms with compact supports

References

1. M. S. Narasimhan, *The problem of limits on a Riemannian manifold*, J. Indian Math. Soc. vol. 20 (1956) pp. 291–297.

2. G. de Rham, *Variétés différentiables*, Paris, Hermann and Cie, 1955.

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