SMALL ISOTOPIES IN EUCLIDEAN SPACES AND 3-MANIFOLDS[1]

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1. **Introduction.** The general type of question considered here is: what homeomorphisms of a space or of a set are obtained by isotopic deformations of a space by a small amount. Although questions of this type have only recently been treated explicitly and for their own sake (e.g. [**2**; **3**; **5**; **6**; **7**]) they had been handled implicitly in work done by Alexander [**1**] and Kneser [**4**] some 35 years ago. In fact this paper owes much to the method of Alexander, rediscovered in a slightly different form.

2. **Definitions.** Let M be a manifold with boundary having a metric d.[2] Denote by $\mathcal{H}(M)$ the set of all homeomorphisms of M onto itself. Define a function ρ of $\mathcal{H}\times\mathcal{H}$ into the extended real number system as follows: $\rho(f, g) = \sup_{x\in M} d(f(x), g(x))$. f and g are ϵ-isotopic if there is an isotopy H_t, $t\in I$, so that $H_0=f$, $H_1=g$ and if t_1, $t_2\in I$ then $\rho(H_{t_1}, H_{t_2})\leqq\epsilon$.

3. **Results.**

THEOREM 1. *If f and g are in $\mathcal{H}(E^n)$ and $\rho(f, g)=\epsilon<\infty$ then f and g are ϵ-isotopic.*

PROOF. By the right invariance of ρ it follows that $\rho(f, g) = \rho(1, gf^{-1})$ and if 1 and gf^{-1} are ϵ-isotopic under H_t then f and g are ϵ-isotopic under H_tf. Hence it suffices to prove the theorem for $f=1$. In this case, using vector notation for points in E^n, let $H_t(x) = tg(x/t)$ for $0<t\leqq 1$ and let $H_0=1$. The continuity of $H_t(x)$ in t and x is clear for $t>0$ and assured for $t=0$ by $d(x, H_t(x)) = td(x/t, g(x/t))\leqq t\epsilon$.

This generalizes to E^n (and slightly strengthens) a recent result of Sanderson for E^3 [**7**]. Alexander's result follows immediately by restricting the isotopy H_t to the unit ball in E^n.

Another direction generalization can take is:

THEOREM 2. *Let M be an arbitrary 3-manifold with boundary having a*

[1] The author is indebted to Professor R. H. Bing for suggesting the problem answered by Theorem 1 for $n=3$, and for directing the thesis, partially summarized here, to which the solution of that problem inexorably led. This research was supported by a Gulf Research and Development Company Fellowship.

[2] All Euclidean spaces E^n will be assumed to have the usual metric. For more general manifolds the metric will be specified as needed.

triangulation Σ. *Let d b the barycentric metric determined by* Σ *and let* ρ *be as defined above. Given any* $\epsilon > 0$ *there is a* $\delta > 0$ *so that if* $f, g \in \mathcal{H}(M)$ *and* $\rho(f, g) < \delta$, *then f and g are* ϵ-*isotopic.*

PROOF. Only a sketch will be given here since the proof is quite long.

Again it suffices to let $f = 1$. The proof is in four stages. If we restrict g to be close to 1 then on each 3-simplex T in Σ we can replace g by g' where $g' \mid \text{Bd } T = 1$ and g' agrees with g except in a small neighborhood of Bd T. An Alexander-type isotopy on each T takes g' onto 1 moving no point far. The global isotopy has the effect of deforming g to a homeomorphism g_1 which is 1 except in a small neighborhood of the 2-skeleton. Next using [6] g_1 is modified to get g_1' which is the same as g_1 on cubes built over the 2-simplexes in Σ and is different from g_1 only near the 1-skeleton of Σ. An isotopy is pieced together again which deforms g_1 to a homeomorphism g_2 which is 1 except in a small neighborhood of the 1-skeleton. Two more reductions, near the 1-skeleton and 0-skeleton respectively, which are described on disjoint cubes near the 1-simplexes and vertices respectively, take g_2 onto the identity.

COROLLARY 1. *If M is a compact* 3-*manifold with boundary, then h is isotopic to* 1 *if and only if* $h = h_k h_{k-1} \cdots h_2 h_1$ *where each* h_i *is the identity outside a polyhedral* 3-*cell.*

COROLLARY 2. *If L is a tame compact* 2-*manifold in any* 3-*manifold M and* $\epsilon > 0$, *there is a* $\delta > 0$ *so that if h is any homeomorphism of L into M moving no point more than* δ *and if* $h(L)$ *is tame, then there is an* ϵ-*isotopy of M taking* $h(L)$ *onto L pointwise and moving no point outside an* ϵ-*neighborhood of L.*

This makes use of and generalizes a result of Sanderson [6].

COROLLARY 3. *If M is a* 3-*manifold having triangulation* Σ *and* $\epsilon > 0$ *there is a* $\delta > 0$ *so that if h is a homeomorphism of the* 2-*skeleton K of* Σ *into M moving no point more than* δ *and such that* $h(K)$ *is tame, then there is an* ϵ-*isotopy of M taking* $h(K)$ *onto K* pointwise.

QUESTION. In Corollary 3 can K be replaced by a 2-complex having no local separating points?

The author has been informed that G. M. Fisher and M. E. Hamstrom separately have obtained Theorem 2 for M a compact 3-manifold with boundary and that the former also obtained Corollary 1.

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