THE ISOMORPHISM THEOREM IN COMPACTLY GENERATED LATTICES

BY PETER CRAWLEY

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It is a well known property of a modular lattice L that if $a, b \in L$ then the quotient sublattices $a \cup b/a$ and $b/a \cap b$ are isomorphic. Morgan Ward [3] has proved that if the ascending or descending chain condition holds, then this property is equivalent to modularity. If the chain conditions are dropped, however, then there are simple examples of nonmodular lattices L for which $a \cup b/a$ and $b/a \cap b$ are isomorphic for all $a, b \in L$.

We shall show here that the isomorphism of all transposed quotients does in general characterize modularity, provided this condition is applied to the ideals of the lattice. More precisely, this result is the following. *If L is an arbitrary lattice, then L is modular if and only if for every pair of ideals A, B of L, the quotients of ideals $A \cup B/A$ and $B/A \cap B$ are isomorphic.*

Actually this result is a corollary of a more general theorem on compactly generated lattices. An element c in a complete lattice L is *compact* if for every subset $S \subseteq L$ with $c \leqq \bigcup S$ there exists a finite subset $S' \subseteq S$ such that $c \leqq \bigcup S'$. A lattice L is said to be *compactly generated* if L is complete and every element of L is a join of compact elements.

THEOREM. *Let L be a compactly generated lattice. If $a \cup b/a$ and $b/a \cap b$ are isomorphic for all $a, b \in L$, then L is modular.*

As the lattice of congruence relations of any algebraic system is compactly generated, compactly generated lattices play a fundamental role in the decomposition theory of algebraic systems, and in this regard the theorem would seem a natural one.[1] It is clear that for any lattice L, the lattice $I(L)$ of all ideals of L is compactly generated. Thus, since $I(L)$ is modular if and only if L is modular, the italicized statement in the second paragraph is a direct consequence of the theorem.

Throughout this note the usual notation and terminology is used. Lattice inclusion, proper inclusion, and covering, are denoted by the symbols $\leqq$, $<$, $\prec$, respectively. The quotient sublattice a/b is defined by $a/b = \{x \in L \mid b \leqq x \leqq a\}$.

[1] For the decomposition theory associated with compactly generated lattices see Dilworth-Crawley [2].

The proof of the theorem requires the following three lemmas.

LEMMA 1. *Let L be a compact generated lattice. If $a, b \in L$ and $a > b$, then there exist elements $p, q \in L$ such that $a \geqq p \succ q \geqq b$.*

LEMMA 2. *Let L be a compactly generated lattice. If $\{x_\alpha\}$ is a chain of elements of L and $a \in L$, then $a \cap \bigcup_\alpha x_\alpha = \bigcup_\alpha a \cap x_\alpha$.*

LEMMA 3. *Let L be a lattice such that $a \cup b/a \cong b/a \cap b$ for all $a, b \in L$. Then if $p, q, r \in L$ and $p \succ q$, either $r \cap p = r \cap q$ or $r \cap p \succ r \cap q$. The dual statement also holds.*

Lemma 1 is given in [**1**, Lemma 2.4], Lemma 2 is essentially given in [**2**, Lemma 2.3], and Lemma 3 is given in [**3**, Lemmas 5.6].

We may now proceed with the proof of the theorem. Suppose L is a compactly generated lattice satisfying the hypothesis of the theorem, but L is not modular. Then L contains a five-element sublattice $\{a, b, t, u, v\}$ such that $a > b$ and $t \cup a = t \cup b = u$, $t \cap a = t \cap b = v$. If a does not cover b, then by Lemma 1 there exist elements $p, q \in L$ such that $a \geqq p \succ q \geqq b$. Clearly $t \cup p = t \cup q = u$ and $t \cap p = t \cap q = v$. Thus the sublattice $\{p, q, t, u, v\}$ is a nonmodular five-element sublattice in which $p \succ q$. Hence we may assume that the sublattice $\{a, b, t, u, v\}$ was originally picked in such a way that $a \succ b$.

Let T be the set of all ordered triples (x, y, z), $x, y, z \in L$, such that $x \geqq a$, $y \geqq b$, $z \geqq v$, and such that the following relations hold:

(i) $t \cup y = u$,

(ii) $t \cap x = z$,

(iii) $x \succ y$,

(iv) $a \cap y = b$.

T is nonempty since the triple (a, b, v) is in T. Now partially order T by defining $(x, y, z) \leqq (x', y', z')$ if and only if $x \leqq x'$, $y \leqq y'$, and $z \leqq z'$. Suppose $\{(x_\alpha, y_\alpha, z_\alpha)\}$ is a chain of T. Let $\bar{x} = \bigcup_\alpha x_\alpha$, $\bar{y} = \bigcup_\alpha y_\alpha$, and $\bar{z} = \bigcup_\alpha z_\alpha$. By infinite associativity and Lemma 2 we have, $t \cup \bar{y} = t \cup \bigcup_\alpha y_\alpha = \bigcup_\alpha t \cup y_\alpha = u$, $t \cap \bar{x} = t \cap \bigcup_\alpha x_\alpha = \bigcup_\alpha t \cap x_\alpha = \bigcup_\alpha z_\alpha = \bar{z}$, and $a \cap \bar{y} = a \cap \bigcup_\alpha y_\alpha = \bigcup_\alpha a \cap y_\alpha = b$. Note that for each index α, $a \cup y_\alpha = x_\alpha$ since $x_\alpha \geqq a$ and $x_\alpha \succ y_\alpha$. Thus $a \cup \bar{y} = a \cup \bigcup_\alpha y_\alpha = \bigcup_\alpha a \cup y_\alpha = \bigcup_\alpha x_\alpha = \bar{x}$. Hence $\bar{x}/\bar{y} = a \cup \bar{y}/\bar{y} \cong a/a \cap \bar{y} = a/b$ by hypothesis, and since $a \succ b$ we must have $\bar{x} \succ \bar{y}$. Thus the triple $(\bar{x}, \bar{y}, \bar{z}) \in T$, and every chain of T has an upper bound. By the Maximal Principle, T contains a maximal element (a_0, b_0, v_0).

Now $u/b_0 = t \cup b_0/b_0 \cong t/t \cap b_0 = t/v_0$, and since $u > a_0 \succ b_0$, there must exist an element $v_1 \in L$ such that $t > v_1 \succ v_0$. Let $a_1 = a_0 \cup v_1$ and b_1

$=b_0 \cup v_1$. We shall show that the triple (a_1, b_1, v_1) satisfies conditions (i)–(iv).

It is clear that $t \cup b_1 = u$. Notice that $a_0 \ngeqq v_1$, since $t \cap a_0 = v_0$. Thus $a_1 = a_0 \cup v_1 \neq a_0 \cup v_0 = a_0$, and hence by Lemma 3, $a_1 \succ a_0$ since $v_1 \succ v_0$. Computing further, $t \cap a_1 \geqq v_1 \succ v_0 = t \cap a_0$, and hence by Lemma 3, $t \cap a_1 \succ v_0$. Thus $t \cap a_1 = v_1$. Just as $a_1 \succ a_0$, it follows that $b_1 \succ b_0$. Hence $a_1 \neq b_1$, for otherwise $b_1 > a_0 > b_0$, contrary to $b_1 \succ b_0$. Thus by Lemma 3 and the fact that $a_0 \succ b_0$, we must have $a_1 \succ b_1$. Finally, consider $a \cap b_1$. Since $a_0 \geqq a > b$ and $b_1 > b$, it follows that $a \cap b_1 \leqq a_0 \cap b_1 = b_0$. Hence $b \leqq a \cap b_1 \leqq a \cap b_0 = b$; whence $a \cap b_1 = b$. Thus the triple (a_1, b_1, v_1) satisfies conditions (i)–(iv), and hence $(a_1, b_1, v_1) \in T$. But (a_1, b_1, v_1) is properly bigger than (a_0, b_0, v_0), and since (a_0, b_0, v_0) was picked to be a maximal element of T, we have a contradiction. Thus L must be modular, and the proof of the theorem is complete.

References

1. P. Crawley, *Lattices whose congruences form a Boolean algebra*, Pacific J. Math. (to appear.)

2. R. P. Dilworth and P. Crawley, *Decomposition theory for lattices without chain conditions*, to appear in Trans. Amer. Math. Soc.

3. Morgan Ward, *A characterization of Dedekind structures*, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 448–451.

California Institute of Technology