ON AN IDENTITY OF BLOCK AND MARSCHAK

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In the Bulletin of the American Mathematical Society H. D. Block and Jacob Marschak proved for each choice of the positive integers \( m \) and \( n \) with \( m \leq n \) the identity

\[
\sum_{1} \left\{ (u_{1} + u_{2} + \cdots + u_{n}) (u_{2} + u_{3} + \cdots + u_{n}) \cdots \right\}
\]

\[
= \left\{ (u_{1} + u_{2} + \cdots + u_{m}) u_{2} u_{3} \cdots u_{n} \right\}^{-1},
\]

where \( u_{1}, \cdots, u_{n} \) denote indefinite numbers and where \( \sum_{1} \) is extended over all the permutations \((s_{1}, s_{2}, \cdots, s_{n})\) of \((1, 2, \cdots, n)\) which rank 1 before each of the numbers \( 2, 3, \cdots, m \).

In this paper I shall prove: If \( p, q \) and \( n \) denote integers with \( 0 \leq p \leq q \leq n \) and \( n \geq 1 \), then

\[
\sum_{2} \left\{ (u_{1} + u_{2} + \cdots + u_{n}) (u_{2} + u_{3} + \cdots + u_{n}) \cdots \right\}
\]

\[
= \left\{ (u_{1} + u_{2} + \cdots + u_{q}) (u_{2} + \cdots + u_{q}) \cdots \right\}^{-1},
\]

where \( \sum_{2} \) is extended over the permutations \((s_{1}, s_{2}, \cdots, s_{n})\) of \((1, 2, \cdots, n)\) which rank 1 before \( 2 \), 2 before \( 3 \), \cdots, \( p-1 \) before \( p \) and finally \( p \) before each of the numbers \( p+1, p+2, \cdots, q \).

The particular case \( p = 1, q = m \) yields (1).

In the proof of (2) I treat first the case \( q = n \). Then \( \sum_{2} \) is extended over the permutations \((s_{1}, \cdots, s_{n})\) with \( s_{h} = h \) \((1 \leq h \leq p)\), where \((s_{p+1}, \cdots, s_{n})\) is an arbitrary permutation of \((p+1, \cdots, n)\). In this case we must show that

\[
\sum_{2} = \left\{ (u_{1} + \cdots + u_{n}) (u_{2} + \cdots + u_{n}) \cdots \right\}^{-1},
\]

In the case \( p = n \) the sum \( \sum_{3} \) consists of only one term namely

\[
\left\{ (u_{1} + \cdots + u_{n}) (u_{n+1} \cdots u_{n}) \right\}^{-1}.
\]

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H. D. Block and Jacob Marschak, *An identity in arithmetic*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 123–124. Without loss of generality we may choose \( i = 1 \) in their identity and then this identity assumes the simpler form indicated in (1).

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(u_1 + \cdots + u_n)^{-1}. I may therefore assume that \( p \) is \( \leq n - 1 \) and that (3) has already been proved with \( p \) replaced by \( p + 1 \). For each integer \( h \geq p + 1 \) and \( \leq n \) the contribution to \( \sum_2^2 \) of the permutations \((s_{p+1}, \ldots, s_n)\) with \( s_{p+1} = h \) is according to the induction hypothesis equal to

\[
u_h \{(u_1 + \cdots + u_n)(u_2 + \cdots + u_n) \cdots (u_{p+1} + \cdots + u_n)u_{p+1} \cdots u_n \}^{-1},
\]

so that

\[
\sum_2 = \{(u_1 + \cdots + u_n)(u_2 + \cdots + u_n) \cdots (u_{p+1} + \cdots + u_n)u_{p+1} \cdots u_n \}^{-1} \sum_{h=p+1}^n u_h,
\]

which gives the required result (3).

Finally we treat the case \( p \leq q \leq n - 1 \) and we may assume that (2) has already been proved with \( p \) replaced by \( p + 1 \). We must prove that

\[
u_{p+1}u_{p+2} \cdots u_n \sum_2 = [p + 1, p + 2, \ldots, q]/[1, 2, \ldots, q],
\]

where

\[
[a_1, a_2, \cdots, a_t] = (u_{a_1} + u_{a_2} + \cdots + u_{a_1})(u_{a_2} + u_{a_3} + \cdots + u_{a_2})
\cdots (u_{a_{t-1}} + u_{a_t})u_{a_t};
\]

the right hand side means 1 if \( t = 0 \).

By the induction hypothesis the contribution to \( u_{p+1} \cdots u_n \sum_2 \) of the permutations \((s_1, s_2, \cdots, s_n)\) which rank \( p \) before \( q+1 \) is equal to \([p + 1, \ldots, q + 1]/[1, \ldots, q + 1]\); the contribution to \( u_{p+1} \cdots u_n \sum_2 \) of the permutations which rank \( q+1 \) between \( h-1 \) and \( h \) is for each integer \( h \) with \( 2 \leq h \leq p \) equal to

\[
u_{q+1}[p + 1, \cdots, q]/[1, \cdots, h - 1, q + 1, h, \cdots, q]
\]

and finally the contribution to \( u_{p+1} \cdots u_n \sum_2 \) of the permutations which rank \( q+1 \) before 1 is equal to

\[
u_{q+1}[p + 1, \cdots, q]/[q + 1, 1, \cdots, q].
\]

In this way we find

\[
u_{p+1}u_{p+2} \cdots u_n \sum_2 = \frac{[p + 1, \cdots, q + 1]}{[1, \cdots, q + 1]}
\]

\[
+ \nu_{q+1} \sum_{h=1}^p \frac{[p + 1, \cdots, q]}{[1, \cdots, h - 1, q + 1, h, \cdots, q]}.
\]
It is therefore sufficient to prove that
\[
\frac{[p + 1, \ldots, g + 1]}{[1, \ldots, q + 1]} + u_{q+1} \sum_{h=1}^{p} \frac{[p + 1, \ldots, q]}{[1, \ldots, h - 1, q + 1, h, \ldots, q]}
= \frac{[p + 1, \ldots, q]}{[1, \ldots, q]}.
\]

This identity is obvious for \( p = 0 \), so that I may assume that \( p \) is \( \geq 1 \) and that (5) has already been proved with \( p \) replaced by \( p - 1 \).

The term with \( h = 1 \) occurring on the left hand side of (5) is equal to
\[
\frac{[p + 1, \ldots, q]}{(u_1 + \cdots + u_q)[2, \ldots, q]},
\]
so that this term is a rational function of \( u_1 \) which possesses at \( u_1 = -(u_2 + \cdots + u_{q+1}) \) a simple pole with residue
\[-\frac{[p + 1, \ldots, q]}{[2, \ldots, q]},\]
and at \( u_1 = -(u_2 + \cdots + u_q) \) a simple pole with residue
\[-\frac{[p + 1, \ldots, q]}{[2, \ldots, q]}.
\]

The left hand side of (5) is therefore a rational function of \( u_1 \) which possesses at \( u_1 = -(u_2 + \cdots + u_{q+1}) \) a simple pole with residue
\[
\frac{[p + 1, \ldots, q + 1]}{[2, \ldots, q + 1]} + u_{q+1} \sum_{h=1}^{p} \frac{[p + 1, \ldots, q]}{[2, \ldots, h - 1, q + 1, h, \ldots, q]}
= \frac{[p + 1, \ldots, q]}{[2, \ldots, q]}.
\]

This expression assumes, if we replace \( u_2, u_3, \ldots, u_{q+1} \) by \( u_1, u_2, \ldots, u_q \), the form
\[
\frac{[p, \ldots, q]}{[1, \ldots, q]} + u_q \sum_{h=1}^{p-1} \frac{[p, \ldots, q - 1]}{[1, \ldots, h - 1, q, h, \ldots, q - 1]}
= \frac{[p, \ldots, q - 1]}{[1, \ldots, q - 1]},
\]
which is equal to zero according to formula (5) applied with \( p \) and \( q \) replaced by \( p - 1 \) and \( q - 1 \). Consequently the left hand side of (5) is a rational function of \( u_1 \) which has at \( u_1 = -(u_2 + \cdots + u_{q+1}) \) a simple pole with residue 0, so that this function is analytic at that
point. This function has at \( u_1 = -(u_2 + \cdots + u_q) \) a simple pole with residue \( [p+1, \cdots, q]/[2, \cdots, q] \) and this is also the case with the function occurring on the right hand side of (5). All the terms occurring in (5) are analytic functions of \( u_1 \), apart of the points \( u_1 = -(u_2 + \cdots + u_{q+1}) \) and \( u_1 = -(u_2 + \cdots + u_q) \), so that the difference between the two sides of (5) is a rational function of \( u_1 \) without poles which tends for \( u_1 \to \infty \) to zero; this difference is therefore identically equal to zero. This completes the proof.

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