1. Introduction. We consider dynamical systems $(X, M)$, where $X$ is a $C^\infty$ vector field on a $C^\infty$ closed manifold $M$ satisfying the following conditions.

1. There are a finite number of singular points of $X$, say $\beta_1, \cdots, \beta_k$, each of simple type. This means that at each $\beta_i$, the matrix of first partial derivatives of $X$ in local coordinates has eigenvalues with real part nonzero.

2. There are a finite number of closed orbits (i.e., integral curves) of $X$, say $\beta_{k+1}, \cdots, \beta_m$, each of simple type. This means that no characteristic exponent (see, e.g., [2]) of $\beta_i, i > k$, has absolute value 1.

3. The limit points of all the orbits of $X$ as $t \to \pm \infty$ lie on the $\beta_i$. In other words, denote by $\phi_t$ the 1-parameter group of transformations generated by $X$ (as we do throughout this paper). Let

$$\alpha(y) = \lim_{t \to -\infty} \phi_t(y), \quad \omega(y) = \lim_{t \to \infty} \phi_t(y), \quad y \in M.$$ 

Then for each $y$, $\alpha(y)$ and $\omega(y)$ are contained in the union of the $\beta_i$.

4. The stable and unstable manifolds of the $\beta_i$ (see §2 for the definition) have normal intersection with each other. More precisely for each $i$ let $W_i$ be the unstable manifold and $W_i^*$ the stable manifold of $\beta_i$ and for $x \in W_i$ (or $W_i^*$) let $W_{ix}$ (or $W_{ix}^*$) be the tangent space of $W_i$ (or $W_i^*$) at $x$. Then for each $i, j$ if $x \in W_i \cap W_j^*$,

$$\dim W_i + \dim W_j^* - n = \dim (W_{ix} \cap W_{jx}^*).$$

See [5] for example for more details.

5. If $\beta_i$ is a closed orbit there is no $y \in M$ with $\alpha(y) = \omega(y) = \beta_i$.

First we remark that systems satisfying (1)–(5) may be very important because of the following possibilities.

(A) It seems at least plausible that systems satisfying (1)–(5) form an open dense set in the space (with the $C^1$ topology) of all vector fields on $M$.

(B) It seems likely that conditions (1)–(5) are necessary and sufficient for $X$ to be structurally stable in the sense of Andronov and Pontrjagin [1]. See also [6].

(A) and (B) have been proved for the case $M$ is a 2-disk, [3] and [9].

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We expect to have more to say about this subject at another time. It is true that conditions (1)–(5) are independent.

With \((X, M)\) as above let \(\sigma_i = \dim W_i\). Then if \(i \leq k\), \(\sigma_i\) is the number of eigenvalues associated to \(\beta_i\) with real part positive. Let \(a_q\) be the number of \(\beta_i, i \leq k\) with \(\sigma_i = q\). If \(i > k\), \(\sigma_i\) is one more than the number of characteristic exponents of \(\beta_i\) with absolute value greater than one. Let \(b_q\) be the number of \(\beta_i, i > k\), with \(\sigma_i = q\).

The main goal of this paper is to prove

**Theorem 1.1.** Let \((X, M)\) be a system satisfying (1)–(5), \(K\) any field, \(R_q\) the rank of \(H^q(M, K)\), and \(M_q = a_q + b_q + b_{q+1}\). Then \(M_q\) and \(R_q\) satisfy the Morse relations

\[
\begin{align*}
M_0 &\geq R_0, \\
M_1 - M_0 &\geq R_1 - R_0, \\
M_2 - M_1 + M_0 &\geq R_2 - R_1 + R_0, \\
&\quad \ldots \ldots \ldots \ldots \\
\sum_{k=0}^{n} (-1)^k M_k &= (-1)^n \chi
\end{align*}
\]

where \(\dim M = n\) and \(\chi\) is the Euler characteristic of \(M\) with respect to \(K\).

Theorem 1.1 contains the Theorem of El'sgol'c [4] which excludes closed orbits. It also contains Reeb's theorem [11] which excludes singular points. However, both El'sgol'c and Reeb made the highly restrictive assumption\(^*\) that no orbit joined saddle points (i.e., \(\beta_i, i \leq k\) with \(\sigma_i \neq 0, n\)) or saddle type closed orbits (i.e., \(\beta_i, i > k\) with \(\sigma_i \neq 1, n\)).

Also it follows from the following theorem which we prove elsewhere that Theorem 1.1 includes the classical theorem of Morse [8] for a function \(f\) on \(M\) with nondegenerate critical points.

**Theorem 1.2.** If \(X = \text{grad } f, f\) a \(C^\infty\) function on \(M\) with nondegenerate critical points, then \(X\) can be \(C^1\) approximated by a \(C^\infty\) field \(Y\) on \(M\) such that \((Y, M)\) satisfies (1)–(5) with no closed orbits.

2. Construction of the stable and unstable manifolds. 2.1. Suppose \(\beta\) is a singular point of simple type of the \(C^\infty\) system \((X, M)\). Let \(k\) be the number of eigenvalues associated to \(\beta\) with real part positive. Then (e.g., [2, p. 330]) there is a \(k\) dimensional \(C^\infty\) submanifold \(W\) of \(M\) passing through \(\beta\) such that if \(x \in W\) then \(\alpha(x) = \beta\). If \(k = 0\), let

\(^*\) Reeb has asked me to note that his footnote 3, 2nd paragraph, of [11, p. 62] (that this assumption is unnecessary) is incorrect.
$W = \beta$. Then $W$ is tangent at $\beta$ to the linear subspace of the tangent space $M_\beta$ of $M$ at $\beta$ defined by these $k$ eigenvalues [2, p. 333]. $W$ is called the unstable manifold of $X$ at $\beta$. Let $R^k$ denote Euclidean $k$-space considered as a vector space. We will show that $W$ is the image of a continuous 1-1 onto map $f: R^k \to W$, with $f(0) = \beta$, and $f$ is $C^\infty$ with Jacobian of rank $k$ except at $0$. Consider the new system $X^*$ obtained by reversing the direction of each vector of $X$ on $M$. Then $\beta$ is a simple singularity of $X^*$ and the above applies to yield the unstable $(n-k)$-dimensional manifold $W^*$ of $X^*$ at $\beta$. Call $W^*$ the stable manifold of $X$ at $\beta$. Note $W$ and $W^*$ have normal intersection at $\beta$.

2.2. Suppose $\beta$ is a closed orbit of $(X, M)$ of simple type. Let $k-1$ be the number of characteristic exponents of $\beta$ with absolute value greater than one. Then ([7] or [13]) there is a $k$-dimensional $C^\infty$ submanifold $W$ of $M$ passing through $\beta$ such that if $x \in W$ then $e^{\alpha(x)} = \beta$. If $k = 1$ let $W = \beta$. Also $W$ is tangent at each point $y$ of $\beta$ to the linear subspace of $M_y$ defined by these $k-1$ characteristic exponents and the tangent vector of $\beta$ at $y$. Call $W$ the unstable manifold of $X$ at $\beta$. We will show there is a continuous 1-1 onto map $f: R^k \to W$, with $f(0) = \beta$, and except along $0 \times S^1$ is $C^\infty$ with Jacobian of rank $k$. Similarly to 2.1, one defines the stable manifold $W^*$ of $X$ at $\beta$ whose dimension in this case is $n-k+1$.

We now construct the map $f$ of 2.1.

There exists a differentiably imbedded $(k-1)$-sphere $K$ in $W$, which is everywhere transversal to $X$. Let $S_0$ be the unit sphere of $R^k$ and $h: S_0 \to K$ be a diffeomorphism. (A diffeomorphism is $C^\infty$ homeomorphism with a differentiable inverse.) Let $\psi_t$ be the 1-parameter group of transformations of $R^k$ generated by the vector field $Y(x) = x$ on $R^k$. For $x \in R^k$, $x \neq 0$, let $t(x)$ be the unique $t$ such that $x/\|x\| = \psi_{t(x)}(x) \in S_0$. Then let $f(0) = \beta$ and $f(x) = \phi_{-t(x)}h\psi_{t(x)}(x)$. It is easy to check that $f: R^k \to W$ thus defined has the desired properties.

To construct the map $f: R^{k-1} \times S^1 \to W$ of 2.2, first let $Y$ be the vector field $(x, 1)$ on $R^{k-1} \times S^1$. Then if $\psi_t$ is the 1-parameter group of transformations generated by $Y$ we have $\psi_t(x, 0) = (xe^t, t \bmod 2\pi)$. Let $R^{k-1} = R^{k-1} \times 0 \subset R^{k-1} \times S^1$, and $C$ be the unit ball in $R^{k-1}$, $\partial C = S_0$. Define $q: R^{k-1} \to R^{k-1}$ by $q(x) = xe^{2\pi}$ and let $q^i S_0 = S_i$ for each integer $i$.

Let $Q$ be a surface of section (i.e., transversal to $X$, see [6]) locally about a point of $\beta$ in $W$, diffeomorphic to a $(k-1)$-cell. Then [6] the orbits of $X$ define a diffeomorphism $k: Q \to Q$ in a neighborhood of $\beta \cap Q$ leaving $\beta \cap Q$ fixed. There is a closed $k$-cell $B$ differentiably
imbedded in \( Q \), \( \partial B = F_0 \) such that \( h^{-1}(F_0) \) is contained in the interior of \( B \). Let \( h^i(F_0) = F_i \), \( i \leq 0 \).

Let \( f \) be an orientation preserving diffeomorphism of a neighborhood \( V_0 \) of \( S_0 \) in \( \mathbb{R}^{k-1} \) into a neighborhood of \( F_0 \) in \( Q \). Then extend \( f \) to a neighborhood of \( \bigcup_{i \leq 0} S_i \) in \( \mathbb{R}^{k-1} \) into a neighborhood of \( \bigcup_{i \leq 0} F_i \) in \( Q \) by the formula

\[
(2.3) \quad f(x) = h^{-1}f_{q_i}(x), \quad x \in \text{nbhd. } V_i \text{ of } S_i.
\]

This makes sense for an appropriate choice of the \( V_i \)'s. Now consider the closed region \( U \) in \( \mathbb{R}^{k-1} \) bounded by \( S_0 \) and \( S_{-1} \). We have defined \( f \) in a neighborhood of the boundary \( \partial U \) of \( U \). After restricting \( f \) to a smaller neighborhood of \( \partial U \), \( f \) can be extended to a diffeomorphism of all of \( U \) into the region of \( Q \) bounded by \( F_0 \) and \( F_{-1} \). This fact follows from arguments which are now standard in differential topology. We won't include them here. Then as in 2.3 we can extend \( f \) to a map of all of \( C \) into \( B \) which is a diffeomorphism except at \( f(0) = \beta \cap Q \).

Next define \( f \) on \( P = \{ \psi \in S, t < 0 \} \) by the following: Let \( \tau(x, \theta) \) be the smallest positive number such that \( \phi_{\tau(x, \theta)}(x, \theta) \) has \( \theta \) as its second coordinate in a fixed product structure \( Q \times \beta, (x, \theta) \in P \). Then let \( f(x, \theta) = \phi_{\tau(x, \theta)}(x, \theta) \). Define \( f: 0 \times S^1 \to \beta \) by \( f(0 \times \theta) = \theta \).

Consider now the surface of section \( S_{-1} \times S^1 \sim A \) in \( \mathbb{R}^{k-1} \times S^1 \) and its image under \( f \). Restrict \( f \) to the closure of the bounded component \( K \) of \( A \). Finally extend \( f \) to all of \( \mathbb{R}^{k-1} \times S^1 \) as follows. For \( y \in \mathbb{R}^{k-1} \times S^1 - K \), let \( t(y) \) be the unique \( t \) such that \( \psi_{t(y)}(y) \in A \). Then let \( f(y) = \phi_{-t(y)}(y) \). After a change of parameter near \( A \), \( f \) will have our desired properties.

3. Implications of (1)–(5). Assume throughout this section that \( (X, M) \) is given as in §1. If \( \beta_i \) is a singular point then \( f_i: R^k \to W_i \) is as in 2.1. If \( \beta_i \) is a closed orbit then \( f_i: \mathbb{R}^{k-1} \times S^1 \to W_i \) is as in 2.2.

**Lemma 3.1.** If \( x \in M, \alpha(x) = \beta_i, \omega(x) = \beta_i, \) then \( \dim W_i \geq \dim W_f \) and equality can occur only if \( \beta_i \) is a closed orbit.

**Proof.** Clearly \( x \in W_i \cap W_j \) and by (4) we have that \( \dim W_i + \dim W_j - n \geq 1 \). But \( \dim W_i^* = n - \dim W_i \) if \( \beta_i \) is a singular point and \( \dim W_j^* = n - \dim W_j + 1 \) if \( \beta_i \) is a closed orbit. Then 3.1 follows.

See [12] for the following.

**Lemma 3.2.** Suppose \( W_i \neq \emptyset \) and \( x \in W_j \). Then there exists a cell neighborhood \( H \) of \( x \) in \( W_j \) such that if \( \delta > 0 \), there is a \( y \in W_i \) with \( d(x, y) < \delta \) and if \( \dim W_i = \dim W_j \), there is a subcell \( K \) of \( W_i \) such that \( H \) and \( K \) are within \( \delta \) in a \( C^1 \) metric.
Define \( \partial W_j = \{ \lim_{k \to \infty} f_j(x_k) \mid x_k \text{ any sequence in } \mathbb{R}^k \text{ with no lps.} \} \). Then let \( \partial^2 W_j = \partial(\partial W_j) \), etc. Note CI \( W_i = W_i \cup \partial W_i \).

**Lemma 3.3.** If \( W_i \cap W_j^* \neq \emptyset \), \( \partial W_i \supset W_j \).

This follows from 3.2.

**Lemma 3.4.** Suppose \( \dim W_i = \dim W_k = \dim W_j \). If \( W_i \cap W_k^* \neq \emptyset \) and \( W_k \cap W_j^* \neq \emptyset \) then \( W_i \cap W_j^* \neq \emptyset \).

**Proof.** Let \( x \in W_k \cap W_j^* \); apply 3.2 using the fact that \( W_i \cap W_k^* \neq \emptyset \). Since \( W_k \) and \( W_j^* \) have normal intersection at \( x \), it follows from 3.2 that \( W_i \cap W_j^* \neq \emptyset \).

**Lemma 3.5.** Suppose \( W_i \cap W_j^* \neq \emptyset \), \( k = 1, \ldots, m \). Then \( W_i \neq W_i^* \) if \( j \neq k \).

**Proof.** First note by 3.1, \( \dim W_{i+k} \leq \dim W_i \) and equality occurs only if \( \beta_{i+k} \) is a closed orbit. This implies we can restrict ourselves to the case of the lemma where all the \( W_i \)'s are of the same dimension. Then if \( W_i = W_j, k \neq j \), 3.4 implies that \( W_i \cap W_j^* \neq \emptyset \). This contradicts condition (5).

**Lemma 3.6.** If \( \partial W_j \cap W_i \neq \emptyset \), then there is a sequence \( W_{i_1}, \ldots, W_{i_n} \) such that \( W_{i_1} \cap W_{i_1+1}^* \neq \emptyset \), \( W_\gamma = W_{i_1} \), and \( W_{i_n} = W_i \).

**Proof.** Let \( \alpha(W_j^*) = \lim_{t \to \infty} W_j^* \). Then it follows that CI \( W_\gamma \cap \alpha(W_j^*) \neq \emptyset \). Let \( \beta_j \in \text{CI } W_j \cap \alpha(W_j^*) \). Then \( W_j \cap W_j^* \neq \emptyset \). If \( j \neq \gamma \), similarly let \( \beta_k \in \text{CI } W_\gamma \cap \alpha(W_j^*) \). Induction and 3.5 yield 3.6.

**Lemma 3.7.** If \( \partial W_j \cap W_i \neq \emptyset \), then \( \partial W_i \supset W_j \) and either \( \dim W_i > \dim W_j \) or \( \dim W_i = \dim W_j \), \( W_i \cap W_j^* \neq \emptyset \), and \( \beta_j \) is a closed orbit.

This follows from 3.6, 3.5, 3.3, 3.1, and 3.4.

**Lemma 3.8.** Each \( W_i \) is an imbedded \( R^p \) or \( R^{p-1} \times S^1 \).

This follows from §2, 3.7 and (5).

**Lemma 3.9.** \( \partial^k W_i \neq \emptyset \), any \( i \) for large enough \( k \).

If not there is a \( W_j \) such that \( \partial^k W_j \cap W_i \neq \emptyset \). By 3.7 then \( \partial W_j \cap W_i \neq \emptyset \), contradicting 3.8.

4. **On Morse theory.** A version of one of the standard theorems of Morse theory is stated in this section. The proof is a short well-known argument using the exact cohomology sequence of a pair and for example can be found in [10].

**Theorem 4.1.** Let \( M \) be an \( n \)-dimensional topological space with closed subspace \( L^p \) for each integer \( p \) such that \( L^p \supset L^{p-1} \), and there
exist integers $a$, $b$ with $L^a = \emptyset$ and $L^b = M$. Using any fixed cohomology theory and coefficient field, assume dimension $H^q(L^p, L^{p-1})$ is finite for each $p$ and $q$. Let $B_q = \dim H^q(M)$ and $M_q = \sum_{p=a}^{b} \dim H^q(L^p, L^{p-1})$. Then $M_q$ and $B_q$ satisfy the Morse relations

$$
M_0 \geq B_0,
M_1 - M_0 \geq B_1 - B_0,
M_2 - M_1 + M_0 \geq B_2 - B_1 + B_0,
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldotted {bad property. It may be that for $W_i \subset K^p$, $\partial W_i$ is not contained in $K^{p-1}$. To avoid this we define a new structure on $M$. We define by induction a sequence of closed subsets $L_i$ of $M$ with $L_i \supset L_{i-1}$ and $L_i = M$ for large enough $i$. Define $L_0 = \emptyset$, and if $L_{i-1}$ has been defined, let $L_i$ be the union of all the $W_j$ whose boundary lies in $L_{i-1}$. It is immediate that $L_i \supset L_{i-1}$, that $L_i$ is closed in $M$ and that $L_i - L_{i-1}$ consists of a disjoint union of $W_j$. It follows from 3.9 that there is an integer $b$ such that $L_b = M$.

One can construct an example to show that the $L_i$ need not be locally connected and that for $W_0 \subset L_i - L_{i-1}$, $W_1 \subset L_{i+1}$, $\dim W_0 = \dim W_1$.

**Lemma 5.1.** Using Čech theory if $M_q$ is as in 1.1 then

$$
M_q = \sum_{i=0}^{b} \dim H^q(L_i, L_{i-1}).
$$

**Proof.** As noted previously $L_i - L_{i-1}$ consists of a disjoint union of the $W_j$ and as $i$ ranges from 0 to $b$ all the $W_j$ are obtained. Denoting cohomology with compact carriers by $K^*_K$, since $H^q_K(P - Q) = H^q(P, Q)$ for Čech theory, we have

$$
\sum_{i=0}^{b} \dim H^q(L_i, L_{i-1}) = \sum_{all \ W_j} \dim H^q_K(W_j).
$$

Using 3.8 and Poincaré duality $H^*_K(W_j) = H_{dim \ W_j} = H^q(W_j)$. Furthermore $\dim H_0(W_j) = 1$ for all $j$, $\dim H_1(W_j) = 1$ if $\beta_j$ is a closed orbit and $\dim H_p(W_j) = 0$ otherwise. The lemma follows.
Theorem 1.1 follows from 4.1 and 5.1.

6. **An analogue of the main theorem.** Suppose instead of a vector field $X$ on $M$, we just have given a $C^\infty$ diffeomorphism $h$ on $M$ which satisfies certain conditions analogous to (1)–(5).

(1') There are a finite number of periodic points (i.e., $x \in M$ such that $h^p(x) = x$ for some integer $p$) of $h$ of simple type (i.e., the differential of $h$ at $p$ has no eigenvalue of absolute value 1).

(3') The limit points of all the orbits of $h$ (i.e., $\{h^p(x) : \text{all integers } p\}$) are periodic points.

(4') The “stable” and “unstable” manifolds of the periodic points have normal intersection.

The previous theory extends to cover this case. In particular if $M_q$ is the number of periodic points with $q$ eigenvalues having absolute value greater than one, the Morse relations of 1.1 hold.

One can ask the corresponding questions of (A) and (B) of §1 for the above situation.

**References**


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