RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A CANONICAL FORM FOR AN ANALYTIC FUNCTION OF SEVERAL VARIABLES AT A CRITICAL POINT[1]

BY N. LEVINSON

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THEOREM. *Let $f(z, w)$ be analytic in (z, w) for small $|z|$ and $|w|$. Let $n > 1$, (since the case $n = 1$ is trivial), let*

$$\frac{\partial^k f}{\partial w^k}(0, 0) = 0 \qquad 1 \leqq k < n,$$

and let

$$\frac{\partial^n f}{\partial w^n}(0, 0) \neq 0.$$

Then there is an analytic function g of (z, s) for small $|z|$ and $|s|$ such that setting

$$w = s + s^2 g(z, s) \tag{1}$$

in $f(z, w)$ yields $f(z, w) = P(z, s)$ where P is a polynomial in s

$$f(z, w) = P(z, s) = \sum_{=0}^{n} p_j(z) s^j. \tag{2}$$

The p_j are analytic for small $|z|$,

$$p_j(0) = 0 \qquad 1 \leqq j \leqq n - 1,$$

and $p_n(0) \neq 0$. Clearly of course (1) implies $s = w + w^2 j(z, w)$ where h is analytic for small $|z|$ and $|w|$. Thus for any small z there is a one to one analytic correspondence between w and s for small $|w|$ and $|s|$.

The result (2) is somewhat reminiscent of the Weierstrass preparation theorem but is different in that the polynomial on the right of (2) is not multiplied by a function of (z, s). On the other hand to achieve the canonical polynomial (2), it is necessary to use the change of variables (1).

A case of this theorem where $n = 3$ arises in the transformation of confluent saddle points to Airy integrals [**1**; **2**] and is treated there.

An indication of the proof of the theorem follows. Since one can take $p_0(z)$ in (2) as $f(z, 0)$ there is no loss of generality in treating the

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case where $f(z, 0)=0$. There is also no restriction in assuming that $\partial^n f/\partial w^n(0, 0)=n!$. Hence it can be assumed that

$$f(z, w) = z^a \sum_1^n b_j(z)w^j + w^n + w^{n+1}\phi(z, w). \tag{3}$$

Here $a>0$ is an integer and ϕ is analytic in (z, w).

If we now introduce

$$w_1 = w[1 + w\phi(z, w)]^{1/n} \tag{4}$$

then for small $|z|$ and $|w|$

$$w_1 = w + c_2(z)w^2 + c_3(z)w^3 + \cdots,$$

and there is an inverse

$$w = w_1 + w_1^2\psi(z, w_1). \tag{5}$$

Using (4) and (5) in (3), $f(z, w)$ becomes

$$f_1(z, w_1) = z^a \sum_1^n b_j(z)[w_1 + w_1^2\psi(z, w)]^j + w_1^n,$$

or

$$f_1(z, w) = z^a \sum_1^n b_j^{(1)}(z)w_1^j + w_1^n + z^a w_1^{n+1}\phi_1(z, w_1), \tag{6}$$

where $\phi_1(z, w_1)$ is analytic for small $|z|$ and $|w_1| \leqq \rho_1$. The proof consists in iterating (4) and showing convergence. Thus the next step involves setting

$$w_2 = w_1[1 + w_1\phi_1(z, w_1)]^{1/n}$$

in (6), giving $f_1 = f_2$ where

$$f_2(z, w_2) = z^a \sum_1^n b_j^{(2)}(z)w_2^2 + w_2^n + z^a w_2^{n+1}\phi_2(z, w_2),$$

The important point is that at each stage ϕ_n comes from terms in f_{n-1} multiplied by z^a. It can be shown that $r_1>0$ can be chosen so that if $|z|<r_1$, the occurrence of $|z|^a \leqq r_1^a$ in ϕ_n at each stage causes convergence finally for $|z|<r_1$ and $|w_1|<\rho_0/2$ where

$$\rho_0 = \rho_1(1 - 1/2)(1 - 1/4)(1 - 1/8)\cdots.$$

A detailed proof will be given elsewhere. The case where z is replaced by several variables is treated in much the same way.

References

1. C. Chester, B. Friedman and F. Ursell, *An extension of the method of steepest descents*, Proc. Cambridge Philos. Soc. vol. 53 (1957) pp. 599–611.

2. B. Friedman, *Stationary phase with neighboring critical points*, J. Soc. Ind. Appl. Math. vol. 7 (1959) pp. 280–289.

Massachusetts Institute of Technology