CLOSED IDEALS IN GROUP ALGEBRAS

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Let \( A(G) \) be the set of all Fourier transforms on the locally compact abelian group \( G \), i.e., the set of all \( f \) of the form

\[
f(x) = \int_{\Gamma} (x, \gamma) F(\gamma) d\gamma \quad (x \in G, F \in L^1(\Gamma)),
\]

where \( \Gamma \) is the dual group of \( G \) and \((x, \gamma)\) is the value of the character \( \gamma \) at the point \( x \). With the norm

\[
\|f\| = \int_{\Gamma} |F(\gamma)| \, d\gamma
\]

\( A(G) \) is a commutative Banach algebra, and \( G \) is its maximal ideal space.

If \( I \) is a closed ideal in \( A(G) \), let \( Z(I) \) be the set of all \( x \in G \) such that \( f(x) = 0 \) for every \( f \in I \). Malliavin [3; 4; 5] has recently solved a problem of long standing by proving that in every nondiscrete \( G \) there is a closed set \( E \) such that \( E = Z(I_1) = Z(I_2) \) for two distinct closed ideals \( I_1 \) and \( I_2 \) in \( A(G) \). Combined with an older result of Helson [1] this implies that there are infinitely many closed ideals \( I \) in \( A(G) \) with \( Z(I) = E \).

It is the purpose of this note to point out that Malliavin’s construction for compact \( G \) (he reduced the general case to this) yields an even more specific result:

**Theorem.** Suppose \( G \) is an infinite compact abelian group. There is a real \( \gamma \in A(G) \) such that the closed ideals \( I_n \) generated by the powers \( f^n \) \((n = 1, 2, 3, \ldots)\) are all distinct.

We sketch the proof. If \( g \in A(G) \) and \( u \) is a real number, we define \( a_\gamma(u) \) by

\[
e^{iug(x)} = \sum_{\gamma \in \Gamma} a_\gamma(u) \cdot (x, \gamma) \quad (x \in G).
\]

Malliavin [5] constructed a real \( g \in A(G) \) for which

\[
|a_\gamma(u)| < \exp(-C|u|^{1/2}) \quad (\gamma \in \Gamma),
\]

where \( C > 0 \) is independent of \( \gamma \). (The exponent \( 1/2 \) in (2) could be

\[\text{81}\]

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replaced by any \( \lambda < 1 \), but not by 1. Kahane's construction [2] should also be mentioned in this connection.) By (2),

\[
\sup_{\gamma \in \Gamma} \int_{-\infty}^{\infty} |a_\gamma(u)u^n| \, du = M_n < \infty \quad (n = 0, 1, 2, \cdots).
\]

The mapping

\[
\phi \rightarrow \int_{\mathcal{G}} \phi(g(x))(-x, \gamma) \, dx
\]

is, for each \( \gamma \), a bounded linear functional in the space of all continuous functions \( \phi \) on the range of \( g \), and hence there are measures \( \mu_\gamma \) on the line, with compact support, such that

\[
\int_{\mathcal{G}} \phi(g(x))(-x, \gamma) \, dx = \int_{-\infty}^{\infty} \phi(t) \, d\mu_\gamma(t).
\]

Taking \( \phi(t) = e^{iut} \), we see that \( a_\gamma(u) \) is the Fourier-Stieltjes transform of \( \mu_\gamma \), and (3) implies that \( d\mu_\gamma(t) = m_\gamma(t) \, dt \), where each \( m_\gamma \) is infinitely differentiable and

\[
|m_\gamma^{(n)}(t)| \leq M_n \quad (\gamma \in \Gamma, t \text{ real}).
\]

Since \( a_0(0) = 1 \), \( m_0 \neq 0 \), and there is a real number \( \alpha \) such that \( m_0(\alpha) \neq 0 \).

Put \( f(x) = g(x) - \alpha \). By (6), the expressions

\[
T_n h = (-1)^n \sum_{\gamma \in \Gamma} H(\gamma)m_\gamma^{(n)}(\alpha) \quad (n = 1, 2, 3, \cdots),
\]

where \( h(x) = \sum H(\gamma)(x, \gamma) \), define bounded linear functionals on \( \mathcal{A}(\mathcal{G}) \). The following two facts show that \( T_n \) annihilates \( I_{n+1} \) but not \( I_n \), and hence establish the theorem:

(A) \( T_nf^n \neq 0 \).

(B) If \( h(x) = (x, \gamma_0)f^{n+1}(x) \), for any \( \gamma_0 \in \Gamma \), then \( T_nh = 0 \).

(A) and (B) are proved by evaluating (7) for all \( h \) of the form

\[
h(x) = P(g(x))(x, -\gamma_0) \quad (\gamma_0 \in \Gamma)
\]

where \( P \) is a polynomial. Set

\[
\epsilon_{j,n}(\gamma) = \int_{-\infty}^{\infty} W_j^{(n)}(t)m_\gamma(t) \, dt,
\]

where \( \{W_j\} \) is a sequence of non-negative infinitely differentiable functions which vanish outside \( (\alpha - 1/j, \alpha + 1/j) \), such that \( \int_{-\infty}^{\infty} W_j(t) \, dt \)
Integrating (9) by parts \( n \) times, we see that \( |c_{j,n}(\gamma)| \leq M_n \) and

\[
\lim_j c_{j,n}(\gamma) = (-1)^n m_{\gamma}(\alpha).
\]

Hence (5) implies, if \( h \) is of the form (8), that

\[
T_n h = \lim_j \sum_{\gamma} H(\gamma) \int_{-\infty}^{\infty} W_j^{(n)}(t)m_{\gamma}(t)dt
\]

\[
= \lim_j \sum_{\gamma} H(\gamma) \int_{\mathbb{G}} W_j^{(n)}(g(x))(x, \gamma)dx
\]

\[
= \lim_j \int_{\mathbb{G}} W_j^{(n)}(g(x))P(g(x))(x, -\gamma_0)dx
\]

\[
= \lim_j \int_{-\infty}^{\infty} W_j^{(n)}(t)P(t)m_{\gamma_0}(t)dt
\]

\[
= (-1)^n \lim_j \int_{-\infty}^{\infty} W_j(t) \left( \frac{d}{dt} \right)^n [P(t)m_{\gamma_0}(t)]dt
\]

\[
= (-1)^n \left( \frac{d}{dt} \right)^n [P(t)m_{\gamma_0}(t)]_{t=\alpha}.
\]

Taking \( h = (g - \alpha)^n \), it follows that \( T_n h \) is the \( n \)th derivative of \((-1)^n (t - \alpha)^n m_{\gamma_0}(t)\), evaluated at \( t = \alpha \), and this is \((-1)^n n! m_{\gamma_0}(\alpha) \neq 0\). This proves (A).

Taking \( h(x) = (x, \gamma_0)(g(x) - \alpha)^{n+1} \), we see that \( T_n h \) is the \( n \)th derivative of \((-1)^n (t - \alpha)^{n+1} m_{\gamma_0}(t)\), evaluated at \( t = \alpha \), which is 0. This proves (B).

References


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