ELLIPTIC PROBLEMS IN WHICH THE BOUNDARY CONDITIONS DO NOT FORM A NORMAL SET

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In most boundary problems for elliptic equations, it is usually assumed that the boundary conditions form a normal set. By this we mean that the highest order derivatives are not permitted to become tangential and that two operators of the same order are not allowed (cf. [1]). The underlying cause is usually the fact that in such cases the adjoint boundary conditions are no longer differential. We shall see, however, that certain general results of the usual theory can be carried over to the more difficult situation.

Let A be a linear elliptic partial differential operator of order $2r$ in a domain G and let $B_1, B_2, \cdots, B_r$ be a set of linear partial differential operators of orders less than $2r$ with coefficients defined on the boundary ∂G of G. The set $\{B_j\}_{j=1}^r$ is called normal if the B_j are of distinct orders and ∂G is nowhere characteristic for any of them. Given such a set $\{B_j\}_{j=1}^r$ which also "covers" A in the sense of [2], it was shown in [3] that one can find another set $\{B_j'\}_{j=1}^r$ such that the boundary problem

$$Au = f \qquad \text{in } G, \tag{1}$$

$$B_j u = 0 \qquad \text{on } \partial G, \qquad j = 1, 2, \cdots, r, \tag{2}$$

has a solution for each f, if, and only if, $v=0$ is the only solution of

$$A^*v = 0 \qquad \text{in } G, \tag{3}$$

$$B_j'v = 0 \qquad \text{on } \partial G, \qquad = 1, 2, \cdots, r. \tag{4}$$

(A^* denotes the formal adjoint of A.)

When the set $\{B_j\}_{j=1}^r$ is not normal, the adjoint conditions (4) are not determined, in general, by partial differential operators alone. One verifies easily that the conditions are integro-differential. We now show how this case can be handled. Let V be the set of smooth functions u which satisfy (2), and let V^* be the set of smooth v satisfying

$$(Au, v) = (u, A^*v)$$

for all $u \in V$, where $(u, v) = \int_G u\bar{v}dx$. In addition, let N be the set of all $u \in V$ for which $Au=0$ in G, and let N^* be the set of all $v \in V^*$ satisfying $A^*v=0$ in G. We now have

THEOREM 1. *If* $(f, N)=0$, *there is a function* $v \in V^*$ *such that* $A^*v=f$ *in* G.

THEOREM 2. *If* $N^*=0$, *then for every* f *there is a function* $u \in V$ *such that* $Au=f$.

The proofs of both these theorems employ the techniques of [3]. Use is made of the fact that the regularity theory of [3] holds also when the set $\{B_j\}_{j=1}^r$ is not normal.

REFERENCES

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