SIMULTANEOUS UNIFORMIZATION[1]

BY LIPMAN BERS

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We shall show that any *two* Riemann surfaces satisfying a certain condition, for instance, any two closed surfaces of the same genus $g>1$, can be uniformized by *one* group of fractional linear transformations (Theorem 1). This leads, in conjunction with previous results [**2**; **3**], to the simultaneous uniformization of *all* algebraic curves of a given genus (Theorems 2–4). Theorem 5 contains an application to infinitely dimensional Teichmüller spaces.

1. Let S be an abstract Riemann surface, f a homeomorphism of bounded eccentricity of S onto another such surface S', and $[f]$ the homotopy class of f. We call $(S, [f], S')$ a *coupled pair* of Riemann surfaces, an *even* (*odd*) pair if f preserves (reverses) orientation. Two coupled pairs, $(S, [f], S')$ and $(S_1, [f_1], S')$ are called *equivalent* if there exist conformal homeomorphisms h and h' with $h(S)=S_1$, $h'(S) =S_1'$, and $[h'fh^{-1}]=[f_1]$.

EXAMPLE. Let m be a Beltrami differential on the Riemann surface S_0, i.e. a differential of type $(-1, 1)$, $m=(\zeta)d\bar{\zeta}/d\zeta$, with $|\mu| \leqq \text{const.} <1$. By S_0^m we denote the surface S_0 with the conformal structure redefined by means of the local metric $|d\zeta+\mu d\bar{\zeta}|$. With m there is associated the even pair $(S_0^m, [1], S)$, where 1 is the identity mapping, and the odd pair $(S_0^m, [\iota], \bar{S}_0)$ where ι denotes the natural mapping of S_0 onto its mirror image $\bar{S}_0$. The latter is defined by replacing each local uniformization ζ on S_0 by $\bar{\zeta}$.

A group G of Möbius transformations will be called *quasi-Fuchsian* if there exists an oriented Jordan curve γ_G (on the Riemann sphere P) which is fixed under G, and if G is fixed-point-free and properly discontinuous in the domains $I(\gamma_G)$ and $E(\gamma_G)$ interior and exterior to γ_G, respectively. If γ_G is a circle, G is a Fuchsian group.

A quasi-Fuchsian group G is canonically isomorphic to the fundamental groups of the two Riemann surfaces $S_1=I(\gamma_G)/G$ and $S_2 =E(\gamma_G)/G$, modulo inner automorphisms. If the resulting isomorphisms of the fundamental groups of S_1 onto those of S_2 can be induced by an orientation reversing homeomorphism f of bounded eccentricity, G is called *proper*. In this case $[f]$ is uniquely determined.

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Thus a proper quasi-Fuchsian group represents a coupled pair $(S_1, [f], S_2)$.

A quasi-Fuchsian group G is said to be of the *first* (*second*) *kind* if the fixed points of elements of G are (are not) dense on γ_G. This is, as one sees at once, a property of S_1 (or of S_2).

2. THEOREM 1. *Let S be a Riemann surface with hyperbolic universal covering surface and $(S, [f], S')$ an odd coupled pair. Then this pair (is equivalent to one which) can be represented by a quasi-Fuchsian group G. If G is of the first kind, then every quasi-Fuchsian group G_1 representing an equivalent coupled pair is of the form $G_1 = QGQ^{-1}$ where Q is a Möbius transformation.*

PROOF. One sees easily that any odd coupled pair is equivalent to one of the form $(S_0^m, [\iota], \bar{S}_0)$; we assume therefore that the given pair already has this form. If S_0^m is not the sphere, the plane, the punctured plane or a torus, the same is true of S_0. In this case the classical uniformization theorem asserts that the pair $(S_0, [\iota], \bar{S}_0)$ can be represented by a Fuchsian group G_0; we may assume that γ_{G_0} is the real axis. There exists a measurable function $\mu(z)$, $|z| < \infty$, such that $\mu(z) \equiv 0$ for $\operatorname{Im} z \leqq 0$ and $\mu(z) d\bar{z}/dz = m$ for $\operatorname{Im} z > 0$. Then $|\mu| \leqq \text{const.} < 1$ and $\mu(z) d\bar{z}/dz$ is invariant under G_0. It is known (cf. for instance, [1]) that there exists a unique solution $\Omega_m(z)$ of the Beltrami equation $\partial\Omega/\partial\bar{z} = \mu(z)\partial\Omega/\partial z$ which has generalized L_2 derivatives and is a homeomorphism of P onto itself, with fixed points at $0, 1, \infty$. If $A_0 \in G_0$, then $\Omega_m A_0$ satisfies the same Beltrami equation; it follows that there is a Möbius transformation A with $\Omega_m A_0 = A\Omega_m$. One verifies easily that $G = \Omega_m G_0 \Omega_m^{-1}$ is a quasi-Fuchsian group representing $(S_0^m, [\iota], \bar{S}_0)$. We note that $\gamma_G = \Omega_m(\gamma_{G_0})$ has two-dimensional measure zero.

Assume next that G is of the first kind and that the quasi-Fuchsian group G_1 represents an equivalent pair. Then there exist conformal mappings ϕ and ψ with $\phi(I(\gamma_G)) = I(\gamma_{G_1})$, $\psi(E(\gamma_G)) = E(\gamma_{G_1})$, $\phi G \phi^{-1} = \psi G \psi^{-1} = G_1$, and, for every $A \in G$, $\phi A \phi^{-1} = B\psi A \psi^{-1} B^{-1}$, B being a fixed element of G_1. Since ψ may be replaced by $B\psi$, we lose no generality in assuming that $B = 1$. The functions $\phi(z)$ and $\psi(z)$ are conformal homeomorphisms between Jordan domains and hence topological on γ_{G_0}. Since $\phi A \phi^{-1} = \psi A \psi^{-1}$ for $A \in G$, we have that $\phi = \psi$ at the fixed points of A. Therefore $\phi = \psi$ on γ_{G_0} and there exists a homeomorphism Q of P onto itself such that $QGQ^{-1} = G_1$, $Q(z) = \phi(z)$ in $I(\gamma_{G_0})$ and $Q(z) = \psi(z)$ in $E(\gamma_{G_0})$.

Using known properties of Ω_m (cf. [1]) and a standard reasoning we verify that $Q\Omega_m$ has L_2 derivatives everywhere; so, therefore, does

Q. Since $\partial Q/\partial \bar{z}=0$ a.e., Q is conformal and hence a Möbius transformation.

3. Consider now a fixed *closed* Riemann surface S_0 of genus $g>1$. The equivalence classes of even coupled pairs $(S, [f], S_0)$ are the points of the *Teichmüller space* T_g. It is known that T_g has a natural complex-analytic structure and can be represented as a *bounded domain* in the number space C^{3g-3}; also T_g is homeomorphic to a cell (cf. [**2**; **3**] and the reference given there). If $\tau=(\tau_1, \cdots, \tau_{3g-3}) \in T_g$, we denote by $(S_\tau, [f_\tau], S_0)$ any pair represented by τ. There exists a properly discontinuous group Γ_g of holomorphic automorphisms of T_g such that S_{τ_1} is conformally equivalent to S_{τ_2} if and only if τ_1 and τ_2 are equivalent under Γ_g.

THEOREM 2. *There exist* $2g$ *Möbius transformations* $A_j^{(\tau)}$ *which depend holomorphically on* $\tau \in T_g$, *satisfy the normalization conditions*: $A_{2g-1}(0)=0$, $A_{2g-1}(\infty)=\infty$, $A_{2g}(1)=1$, $\prod_{j=1}^{g} A_{2j-1}A_{2j}A_{2j-1}^{-1}A_j^{-1}=1$, *and generate, for each fixed* τ, *a quasi-Fuchsian group* G_τ *with* $I(\gamma_{G_\tau})/G_\tau$ *conformally equivalent to* S_τ.

Holomorphic dependence of $A^{(\tau)}$ on $\tau \in T_g$ means, of course, that $A^{(\tau)}(z)=[a(\tau)z+b(\tau)]/[c(\tau)z+d(\tau)]$, where a, b, c, d are holomorphic functions.

SKETCH OF PROOF. We may assume that $0 \in T_g$ corresponds to the pair $(S_0, [1], S_0)$. Let G_0 be the Fuchsian group (with γ_{G_0} the real axis) representing the odd pair $(S_0, [\iota], \bar{S}_0)$, and let $\{A_1^{(0)}, \cdots, A_{2g}^{(0)}\}$ be a suitably normalized set of generators of G_0. Every pair $(S_\tau, [f_\tau], S_0)$ is equivalent to one of the form $(S_0^m, [1], S_0)$. Let Ω_m be as in the proof of Theorem 1, and set $A_j^{(\tau)}=\Omega_m A_j^{(0)}\Omega_m^{-1}$. Using Theorem 1 and the properties of Ω_m proved in [1], as well as the definition of the complex analytic structure of T_g (cf. [**2**]), one verifies that $A_j^{(\tau)}$ depend only on τ and not on m, and have the required properties.

Note that $\gamma_G=\Omega_m(\gamma_{G_0})$, so that this curve admits the representation $z=\sigma(t, \tau)$, $-\infty<t<+\infty$, where σ depends holomorphically on τ, and $\sigma\to\infty$ for $|t|\to\infty$.

4. Next, let S_0 be as before, and let S_1 denote the surface obtained by removing some fixed point from S_0. The equivalence classes of even coupled pairs $(S, [f], S_1)$ are the points of the Teichmüller space $T_{g,1}$ which is again a complex manifold homeomorphic to a cell and representable as a bounded domain in C^{3g-2}. Using the methods of the proof of Theorem 2 it is not difficult to establish.

THEOREM 3. *$T_{g,1}$ is holomorphically equivalent to the domain $M_{g,1} \subset C^{3g-2}$ defined as follows*: $(z, \tau) = (z, \tau_1, \cdots, \tau_{3g-3}) \in M_{g,1}$ *if and only if* $\tau \in T_g$ *and* $z \in I(\gamma_{G_\tau})$.

The results of [2, §10] can now be restated as

THEOREM 4. *There exist finitely many meromorphic functions,* $F_1(z, \tau), \cdots, F_N(z, \tau)$, *in* $M_{g,1}$ *which, for every fixed* τ, *generate the field of automorphic functions in* $I(\gamma_{G_\tau})$ *under the group* G_τ, *i.e.—the field of meromorphic functions on* S_τ.

These functions uniformize simultaneously all algebraic function fields of genus g, just as the functions $\wp(z, 1, \tau)$, $\wp'(z, 1, \tau)$, $|z| < \infty$, $\text{Im}\, \tau > 0$, uniformize all elliptic function fields.

5. Finally let S_0 be any open Riemann surface without nontrivial conformal self-mapping homotopic to the identity. The Teichmüller space $T(S_0)$, i.e. the space of equivalence classes of even pairs $(S, [f], S_0)$ is a complete metric space (under the Teichmüller distance) but, in general, infinitely dimensional. Nevertheless we may define a continuous complex valued function Φ on $T(S_0)$ to be *holomorphic* if for every $p_1 = (S_1, [f], S_0) \in T(S_0)$ and every finite sequence $(m_1, \cdots, m_r)$ of Beltrami's differentials on S_1, the mapping of a neighborhood of $0 \in C_r$ into C given by

$$(\zeta_1, \cdots, \zeta_r) \rightarrow p = (S_1^{\zeta_1 m_1 + \cdots + \zeta_r m_r}, [f], S_0) \rightarrow \Phi(p)$$

is holomorphic. The method of proof of Theorem 2 yields

THEOREM 5. *If* $(S_0, [\iota], \bar{S}_0)$ *is representable by a Fuchsian group of the first kind, then there exist a finite or infinite sequence of Möbius transformations* $\{A_j^{(P)}\}$, *depending holomorphically on* $p \in T(S_0)$ *and such that, for every fixed* $q = (S_1, [f], S_0) \in T(S_0)$, *the* $A_j^{(q)}$ *generate a quasi-Fuchsian group* G_q *with* $I(\gamma_{G_q})/G_q$ *conformally equivalent to* S_1.

Thus there are many holomorphic functions as $T(S_0)$, in particular, enough functions to *separate points*.

REFERENCES

1. L. V. Ahlfors and Lipman Bers, *Riemann's mapping theorem for variable metrics*, (to appear).

2. Lipman Bers, *Spaces of Riemann surfaces*, Proceedings of the International Congress of Mathematicians, Edinburgh, 1958, pp. 309–361.

3. ———, *Spaces of Riemann surfaces as bounded domains*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 98–103.

NEW YORK UNIVERSITY