Theorems of Jackson and S. Bernstein about the approximation of smooth functions are usually interpreted in the way that all functions with a prescribed degree of smoothness have a definite degree of approximation. They can be viewed in another way, which reveals their susceptibility to generalization.

Let \( \omega(h) \) be an increasing continuous subadditive function defined for \( h \geq 0 \) with \( \omega(0) = 0 \), \( A \) a compact metric space with infinitely many points. By \( C_1^\omega \) we denote the set of all real valued functions \( f \) on \( A \) with \( |f(x)| \leq 1 \), \( |f(x) - f(x')| \leq \omega(h) \), \( h = \rho(x, x') \). If \( A \) is a \( q \)-dimensional cube, \( \rho \) a natural number and \( 0 < \alpha \leq 1 \), we denote by \( C_1^{q+\alpha} \) the set of all functions on \( A \) with continuous partial derivatives of orders not exceeding \( \rho \) and bounded by 1, and with the derivatives of order \( \rho \) satisfying a Lipschitz condition of order \( \alpha \) and with coefficient 1. Let \( G = \{g_n\} \) be a sequence of continuous functions on \( A \). Then, with some norm, for example the uniform norm on \( A \),

\[
E_n(f) = E_n^\omega(f) = \inf \left\{ \left\| f - \sum_{i=1}^{n} a_i g_i \right\| \right\}
\]

is the degree of approximation of \( f \) by linear combinations of \( g_1, \ldots, g_n \); and

\[
\varepsilon_n(W) = \sup_{f \in W} E_n(f)
\]

is the degree of approximation of a class \( W \).

The theorems of Jackson and Bernstein state that for periodic \( f \in C_1^{q+\alpha} \), and the trigonometric approximation, \( E_n(f) \) has the exact order \( n^{-(q+\alpha)/\alpha} \); exceptions occur only if \( f \) has a higher degree of smoothness. We regard this as a statement about a certain massivity of \( C_1^{q+\alpha} \), which prevents better approximation by linear combinations of only \( n \) functions. One can hope that an estimate of \( \varepsilon_n(C_1^{q+\alpha}) \) from below can be given for an arbitrary system \( G \), and that the trigonometric system is close to the best possible. That this is true, is shown by the following results:

**Theorem 1.** Let \( A \) be a compact metric space, and \( \delta = \delta(n) \) the largest

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number such that there exist \( n \) points of \( A \) with mutual distances \( \geq \delta \). Then for each \( G \),

\[
\varepsilon_n(C_1^n) \geq \frac{1}{2} \omega(\delta(n + 1)).
\]

This cannot be essentially improved, for there exists a \( G \) with \( \varepsilon_n(C_1^n) = \omega(\delta_1) \), if \( A \) can be covered by \( n \) open balls of radius \( \delta_1 = \delta_1(n) \).

**Theorem 2.** If \( A \) is a \( q \)-dimensional cube, then for some constant \( B \), and each \( G \) in the uniform and the \( L^1 \) norm

\[
(1) \quad \varepsilon_n(C_1^{p+\alpha}) \geq Bn^{-(p+\alpha)/q}, \quad p = 0, 1, \ldots; 0 < \alpha < 1.
\]

From these and similar theorems one can obtain by a method of condensation of singularities:

**Theorem 3.** If \( A \) is as in Theorem 2, \( p = 0, 1, \ldots \) and \( 0 < \alpha < 1 \), then there exists a constant \( B \) such that for each system \( G \) one can find a function \( f_0(z) \) such that, in the uniform and the \( L^1 \) norm,

\[
E_n(f_0) \geq Bn^{-(p+\alpha)/q}
\]

for an infinite number of values of \( n \).

**Theorem 4.** Let \( A_p \) be the ellipse with focii \(-1, +1\) and the sum of the half-axes \( 2p \). For each \( G \) and each sequence \( \varepsilon_n \to 0 \) there exists a function \( f_0(z) \), analytic inside \( A_p \), with \( |f_0(z)| \leq 1 \) such that the degree of approximation of \( f_0 \) on \((-1, +1)\) satisfies, in the \( L^1 \) norm,

\[
(2) \quad E_n(f_0) \geq \varepsilon_n^{p^{-n}}
\]

for infinitely many \( n \).

Similar and more general results were recently obtained by A. G. Vituškin [1]. However, his lower bounds (for \( s = n+1, m = 0 \) in [1]) are of orders \( (n \log n)^{-\alpha/(p+\alpha)q} \) and \( n^{-\alpha n}, \alpha > 0 \), for the problems of types (1), and (2), respectively.

**Reference**