LIFTING THE ACTION OF A GROUP IN A FIBRE BUNDLE

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1. Suppose that B is a G-space for a given topological group G. That is we are given a continuous map $\bar{\alpha}: G\times B \to B$ satisfying the equations

$$\bar{\alpha}(u_1 \cdot u_2, b) = \bar{\alpha}(u_1, \bar{\alpha}(u_2, b)), \qquad u_1, u_2 \in G, b \in B,$$
$$\bar{\alpha}(e, b) = b, \qquad e \text{ the identity of } G.$$

Let $\mathfrak{B}$ be a principal bundle over B, [5, p. 35] with total space E and H the structural group so that B may be regarded as the orbit space of E by H. We wish to consider here the problem of putting the two actions together in E in a sense to be made precise below.

2. Let $\mathfrak{B}$ be a bundle with base B and total space E. Suppose B is a G-space given by a function $\bar{\alpha}$ as above. We say that action $(G, \bar{\alpha})$ can be lifted to E in $\mathfrak{B}$ if E can be given the structure of a G-space so that the projection of E onto B in $\mathfrak{B}$ is an equivariant map, i.e. so that if α gives the action of G on E we have the following commutative diagram:

$$\begin{array}{ccc} G\times E & \xrightarrow{\alpha} & E \\ (1, p)\downarrow & & \downarrow p \\ G \times B & \underset{\alpha}{\rightarrow} & B \end{array} \tag{2.1}$$

(G, α) will then be called a lifting of the action $(G, \bar{\alpha})$. A lifting will be called a bundle lifting in $\mathfrak{B}$ if for each $u \in G$ the map $x \to \alpha(u, x)$ of E onto E is a bundle mapping.

For example, a group of diffeomorphisms of a manifold B in the C^r-topology has a bundle lifting in the tangent bundle to B in taking the differential of each element.

PROPOSITION 2.1. *If the action $(G, \bar{\alpha})$ on B has a bundle lifting in the principal bundle $\mathfrak{B}$ with structural group H and total space E, then $G\times H$ acts on E in a canonical way. If $(G, \bar{\alpha})$ is a transitive action so is the action of $G\times H$. If the action $(G, \bar{\alpha})$ is free so is that of $G\times H$.*

If (G, α) is the bundle lifting in B of $(G, \bar{\alpha})$ define the action $(G\times H, \beta)$ in E by $\beta((u, h), x) = \alpha(u, x)\cdot h$.

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PROPOSITION 2.2. *Let* $\mathcal{B}$ *be a principal bundle over a topological group* G. *The action of* G *on itself by left translation can be lifted to* E *in* B *if and only if* $\mathcal{B}$ *is the product bundle.*

Indeed, each orbit of G in E will be a cross-section. Since there are nontrivial bundles over almost all topological groups, this shows that the lifting in general is impossible. However, a measure of what stops the lifting will then be a measure of the nontrivialness of the bundle in this case. Similarly, in the case of Proposition 2.1 such a measure would tell us those bundles not obtained in the canonical way of factoring out a suitable isotropy group in a transitive action.

3. This section gives the statement and sketches the proof of the main theorem. We assume throughout this section that $\mathcal{B}$ is a principal bundle with structural group H, a torus of dimension m. We also assume that G is a semi-simple, compact, connected Lie group. To avoid coverings we suppose G is simply connected. Nothing is lost in this last assumption as we have not demanded that the action of G be effective.

THEOREM 3.1. *Under the above hypothesis, if* B *is paracompact and satisfies the first countability axiom then there is a bundle lifting of* $(G, \bar{\alpha})$.[2]

LEMMA 3.2. *In order that there be a bundle lifting under our hypothesis, it is sufficient that we have a bundle lifting over a fixed neighborhood* U *of the identity in* G.

This follows from the monodromy theorem [1, p. 49].

LEMMA 3.3. *The mapping* $\bar{\alpha}$ *restricted to* $U\times B$ *where* U *is a suitably chosen neighborhood of* e *in* G *can be lifted to a bundle map* $\alpha\colon U\times E\rightarrow E$ *such that* $\alpha(e, x)=x$, $x\in E$.

Choose U homeomorphic to a cube and apply the covering homotopy theorem, [4, p. 555]. (The paracompactness is needed here.)

DEFINITION 3.1. Let V be a neighborhood of e in G with $V^3\subset U$. We define the error function $\bar{f}\colon V\times V\times B\rightarrow H$ by the equation

$$\alpha(u_1\cdot u_2, x) = \alpha(u_1, \alpha(u_2, x))\cdot\bar{f}(u_1, u_2; p(x)), \qquad x\in E \tag{3.1}$$

where α is given by Lemma 3.3. $\bar{f}$ is clearly continuous.

From the associative law we have

[2] The author wishes to express his appreciation to Dr. R. S. Palais for pointing out that a hypothesis of simple connectivity is not needed.

$$(3.2)\quad \bar{f}(u_1, u_2; u_3 b)^{-1}\cdot\bar{f}(u_2, u_3; b)\cdot\bar{f}(u_1, u_2, u_3; b)\cdot\bar{f}(u_1\cdot u_2, u_3; b)^{-1} = 1,$$

where we use the notation $u\cdot b=\bar{\alpha}(u, b)$. If $\bar{g}$: $V\times B\to H$ is a continuous map then we define a new covering α' by

$$(3.3)\qquad \alpha'(u, x) = \alpha(u, x)\cdot\bar{g}(u; x).$$

In order that α' give the action of a local group it is necessary and sufficient that we have

$$(3.4)\qquad \bar{f}(u_1, u_2; b) = \bar{g}(u_2; b)\cdot\bar{g}(u_1; u_2\cdot b)\cdot\bar{g}(u_1\cdot u_2; b)^{-1}.$$

Taking V contractible, we see that the map $\bar{f}$ is homotopic to the constant map $V\times V\times B\to 1\in H$. Then there is a unique lifting of $\bar{f}$ to a map f: $V\times V\times B\to E^m$, euclidean m-space, satisfying $f(e, e, b_0)=0$. Then f will satisfy the equation (3.2) in E^m, (with the notation changed to additive notation). It is sufficient then to find g: $V\times B\to E^m$ satisfying (3.4) for f in place of $\bar{f}$. We make now the following conversions.

$$(3.5)\qquad P(u_1, u_2, u_3)(b) = f(u_1^{-1}u_2, u_2^{-1}u_3; u_3^{-1}\cdot b)$$

then (3.2) becomes

$$(3.6)\qquad \begin{aligned} &P(u_2, u_3, u_4)(b) - P(u_1, u_3, u_4)(b) + P(u_1, u_2, u_4)(b)\\ &\qquad\qquad\qquad - P(u_1, u_2, u_3)(b) = 0.\end{aligned}$$

Furthermore it is easily seen that

$$(3.7)\qquad \begin{aligned} P(uu_1, uu_2, uu_3)(ub) &= P(u_1, u_2, u_3)(b),\\ P(e, e, e)(b) &= 0.\end{aligned}$$

We consider P as a continuous function defined in a neighborhood of the diagonal Δ in $G\times G\times G$ and taking values in the topological group $(E^m)^B$ of continuous functions of B into E^m in the compact open topology. This group is easily seen to be an AR-space. Now suppose we have a continuous function Q defined on some neighborhood of the diagonal in $G\times G$ and satisfying

$$(3.8)\qquad P(u_1, u_2, u_3) = Q(u_2, u_3) - Q(u_1, u_3) + Q(u_1, u_2),$$

$$(3.9)\qquad Q(uu_1, uu_2)(ub) = Q(u_1, u_2)(b).$$

Then if we define $g(u; b)=Q(e, u)(u\cdot b)$ g will satisfy 2.4. g is simultaneously continuous in both variables since B satisfies the first axiom of countability [**3**, p. 103], and we could then conclude that the desired lifting exists.

We might as well assume that P is defined on all of $G\times G\times G$ since

$(E^m)^B$ is an AR space. Now consider the sheaf of germs of continuous Alexander-Spanier cochains of degree n with coefficients in $(E^m)^B$. This sequence of sheaves forms a soft, (since $(E^m)^B$ is an AR space), acyclic resolution of the simple sheaf over G with coefficients in $(E^m)^B$. By the Cartan uniqueness theorem, [**2**, p. 181] it follows that it yields the same cohomology for G as does the usual Alexander Spanier cohomology. Since $H^2(G; (E^m)^B)=0$ in this last theory it follows that there exists a continuous function $Q: G\times G\to(E^m)^B$ which satisfies (3.8) in a neighborhood of the diagonal Δ in $G\times G$. Since G is compact we might as well assume this neighborhood homogeneous. Define

$$Q(u_1, u_2)(b) = \int_G Q(uu_1, uu_2)(ub)du$$

where the integral is the usual normalized Haar measure. Q is the desired cochain. It also can be shown that the lifting is unique up to a bundle equivalence.

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