LIFTING THE ACTION OF A GROUP IN A FIBRE BUNDLE

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Communicated by Deane Montgomery, January 8, 1960

1. Suppose that \( B \) is a \( G \)-space for a given topological group \( G \). That is we are given a continuous map \( \alpha : G \times B \to B \) satisfying the equations

\[
\alpha(u_1 \cdot u_2, b) = \alpha(u_1, \alpha(u_2, b)), \quad u_1, u_2 \in G, \ b \in B, \\
\alpha(e, b) = b, \quad e \text{ the identity of } G.
\]

Let \( \mathfrak{B} \) be a principal bundle over \( B \), [5, p. 35] with total space \( E \) and \( H \) the structural group so that \( B \) may be regarded as the orbit space of \( E \) by \( H \). We wish to consider here the problem of putting the two actions together in \( E \) in a sense to be made precise below.

2. Let \( \mathfrak{B} \) be a bundle with base \( B \) and total space \( E \). Suppose \( B \) is a \( G \)-space given by a function \( \alpha \) as above. We say that action \((G, \alpha)\) can be lifted to \( E \) in \( \mathfrak{B} \) if \( E \) can be given the structure of a \( G \)-space so that the projection of \( E \) onto \( B \) in \( \mathfrak{B} \) is an equivariant map, i.e. so that if \( \alpha \) gives the action of \( G \) on \( E \) we have the following commutative diagram:

\[
\begin{array}{ccc}
G \times E & \xrightarrow{\alpha} & E \\
\downarrow & & \downarrow \beta \\
G \times B & \xrightarrow{\alpha} & B
\end{array}
\]

\((G, \alpha)\) will then be called a lifting of the action \((G, \alpha)\). A lifting will be called a bundle lifting in \( \mathfrak{B} \) if for each \( u \in G \) the map \( x \mapsto \alpha(u, x) \) of \( E \) onto \( E \) is a bundle mapping.

For example, a group of diffeomorphisms of a manifold \( B \) in the \( C^r \)-topology has a bundle lifting in the tangent bundle to \( B \) in taking the differential of each element.

**Proposition 2.1.** If the action \((G, \alpha)\) on \( B \) has a bundle lifting in the principal bundle \( \mathfrak{B} \) with structural group \( H \) and total space \( E \), then \( G \times H \) acts on \( E \) in a canonical way. If \((G, \alpha)\) is a transitive action so is the action of \( G \times H \). If the action \((G, \alpha)\) is free so is that of \( G \times H \).

If \((G, \alpha)\) is the bundle lifting in \( B \) of \((G, \alpha)\) define the action \((G \times H, \beta)\) in \( E \) by \( \beta((u, h), x) = \alpha(u, x) \cdot h. \)

1 The author holds a National Science Foundation Postdoctoral Fellowship.
PROPOSITION 2.2. Let $\mathcal{B}$ be a principal bundle over a topological group $G$. The action of $G$ on itself by left translation can be lifted to $E$ in $B$ if and only if $\mathcal{B}$ is the product bundle.

Indeed, each orbit of $G$ in $E$ will be a cross-section. Since there are nontrivial bundles over almost all topological groups, this shows that the lifting in general is impossible. However, a measure of what stops the lifting will then be a measure of the nontrivialness of the bundle in this case. Similarly, in the case of Proposition 2.1 such a measure would tell us those bundles not obtained in the canonical way of factoring out a suitable isotropy group in a transitive action.

3. This section gives the statement and sketches the proof of the main theorem. We assume throughout this section that $\mathcal{B}$ is a principal bundle with structural group $H$, a torus of dimension $m$. We also assume that $G$ is a semi-simple, compact, connected Lie group. To avoid coverings we suppose $G$ is simply connected. Nothing is lost in this last assumption as we have not demanded that the action of $G$ be effective.

THEOREM 3.1. Under the above hypothesis, if $B$ is paracompact and satisfies the first countability axiom then there is a bundle lifting of $(G, \tilde{\alpha})$.

LEMMA 3.2. In order that there be a bundle lifting under our hypothesis, it is sufficient that we have a bundle lifting over a fixed neighborhood $U$ of the identity in $G$.

This follows from the monodromy theorem [1, p. 49].

LEMMA 3.3. The mapping $\tilde{\alpha}$ restricted to $U \times B$ where $U$ is a suitably chosen neighborhood of $e$ in $G$ can be lifted to a bundle map $\alpha: U \times E \to E$ such that $\alpha(e, x) = x$, $x \in E$.

Choose $U$ homeomorphic to a cube and apply the covering homotopy theorem, [4, p. 555]. (The paracompactness is needed here.)

DEFINITION 3.1. Let $V$ be a neighborhood of $e$ in $G$ with $V^2 \subseteq U$. We define the error function $\tilde{J}: V \times V \times B \to H$ by the equation

$$\alpha(u_1 \cdot u_2, x) = \alpha(u_1, \alpha(u_2, x)) \cdot \tilde{J}(u_1, u_2; p(x)), \quad x \in E$$

where $\alpha$ is given by Lemma 3.3. $\tilde{J}$ is clearly continuous.

From the associative law we have

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The author wishes to express his appreciation to Dr. R. S. Palais for pointing out that a hypothesis of simple connectivity is not needed.
where we use the notation $u \cdot b = \alpha(u, b)$. If $g: V \times B \to H$ is a continuous map then we define a new covering $\alpha'$ by

$$
(3.3) \quad \alpha'(u, x) = \alpha(u, x) \cdot g(u; x).
$$

In order that $\alpha'$ give the action of a local group it is necessary and sufficient that we have

$$
(3.4) \quad f(u_1, u_2, b) = g(u_2, b) - g(u_1, b) - g(u_1, u_2, b)^{-1}.
$$

Taking $V$ contractible, we see that the map $f$ is homotopic to the constant map $V \times V \times B \to \{b \in H \}$, then there is a unique lifting of $f$ to a map $\tilde{f}: V \times V \times B \to E^m$, euclidean $m$-space, satisfying $f(e, e, b_0) = 0$. Then $\tilde{f}$ will satisfy the equation (3.2) in $E^m$, (with the notation changed to additive notation). It is sufficient then to find $g: V \times B \to E^m$ satisfying (3.4) for $f$ in place of $\tilde{f}$. We make now the following conversions.

$$
(3.5) \quad P(u_1, u_2, u_3)(b) = f(u_1^{-1}u_2, u_2^{-1}u_3; u_3^{-1}b)
$$

then (3.2) becomes

$$
(3.6) \quad P(u_2, u_3, u_4)(b) - P(u_1, u_3, u_4)(b) + P(u_1, u_2, u_4)(b)
\quad - P(u_1, u_2, u_3)(b) = 0.
$$

Furthermore it is easily seen that

$$
(3.7) \quad P(uu_1, uu_2, uu_3)(ub) = P(u_1, u_2, u_3)(b),
\quad P(e, e, e)(b) = 0.
$$

We consider $P$ as a continuous function defined in a neighborhood of the diagonal $\Delta$ in $G \times G \times G$ and taking values in the topological group $(E^m)^B$ of continuous functions of $B$ into $E^m$ in the compact open topology. This group is easily seen to be an $AR$-space. Now suppose we have a continuous function $Q$ defined on some neighborhood of the diagonal in $G \times G$ and satisfying

$$
(3.8) \quad P(u_1, u_2, u_3) = Q(u_2, u_3) - Q(u_1, u_3) + Q(u_1, u_2),
\quad Q(\mu u_1, \mu u_2)(ub) = Q(u_1, u_2)(b).
$$

Then if we define $g(u; b) = Q(e; u)(u \cdot b)$ $g$ will satisfy 2.4. $g$ is simultaneously continuous in both variables since $B$ satisfies the first axiom of countability [3, p. 103], and we could then conclude that the desired lifting exists.

We might as well assume that $P$ is defined on all of $G \times G \times G$ since
$(E^m)^B$ is an $AR$ space. Now consider the sheaf of germs of continuous Alexander-Spanier cochains of degree $n$ with coefficients in $(E^m)^B$. This sequence of sheaves forms a soft, (since $(E^m)^B$ is an $AR$ space), acyclic resolution of the simple sheaf over $G$ with coefficients in $(E^m)^B$. By the Cartan uniqueness theorem, [2, p. 181] it follows that it yields the same cohomology for $G$ as does the usual Alexander Spanier cohomology. Since $H^2(G; (E^m)^B) = 0$ in this last theory it follows that there exists a continuous function $Q: G \times G \rightarrow (E^m)^B$ which satisfies (3.8) in a neighborhood of the diagonal $\Delta$ in $G \times G$. Since $G$ is compact we might as well assume this neighborhood homogeneous. Define

$$Q(u_1, u_2)(b) = \int_G Q(uu_1, uu_2)(ub)du$$

where the integral is the usual normalized Haar measure. $Q$ is the desired cochain. It also can be shown that the lifting is unique up to a bundle equivalence.

**References**