GROUPS OF AUTOMORPHISMS OF ALMOST KAEHLER MANIFOLDS

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1. Let M be a compact almost Kaehler manifold[2] of real dimension $2n$. The fundamental 2-form ω (which together with the metric g of M defines the almost Kaehlerian structure) is harmonic and therefore invariant by every infinitesimal isometry [2]. Let X be an infinitesimal conformal transformation of M. Then, for all $n>1$ we shall show that X is in fact an infinitesimal isometry. Indeed, the following theorem is proved:

THEOREM 1. *The largest connected Lie group of conformal transformations of a compact almost Kaehler manifold* M^{2n} $(n>1)$ *coincides with the largest connected group of automorphisms of the almost Kaehlerian structure. Moreover, the infinitesimal automorphisms are infinitesimal isometries.*

This generalizes a previous result [3]. If the almost complex structure is completely integrable and comes from a complex analytic structure we obtain the theorem of Lichnerowicz [5] whose methods it seems cannot be extended to include the almost Kaehler manifolds.

In the noncompact case if we consider infinitesimal conformal maps whose covariant forms are closed, a much wider class of manifolds may be considered.

2. Let X be a vector field on M whose image by the almost complex structure operator J is "closed," that is, its covariant form $C\xi$ is closed where C is the complex structure operator applied to forms. Then, X is an infinitesimal automorphism of M. Denote by $t(X)$ the tensorfield $\theta(X)J$ modulo $i(X)D\omega$ where $\theta(X)$, $i(X)$ and D are the Lie derivative, interior product and covariant differential operators, resp. For Kaehler manifolds $D\omega$ vanishes, and so $t(X)$ and $\theta(X)J$ coincide. In this case, the vanishing of $t(X)$ characterizes the infinitesimal analytic transformations. Let t be a covariant real tensor of order 2 and denote by J again the operator

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[2] The manifolds, differential forms and tensor fields considered are assumed to be of class C^∞.

$$J: t_{ij} \longrightarrow t_{ir} w^r_j, \qquad \omega = w_{ij} dx^i \wedge dx^j.$$

Clearly, $J\omega = g$. For any vector field X on a Kaehler manifold it can be shown that

$$\bar{\theta}(X)\omega - \theta(X)\omega = \delta\xi \cdot \omega - 2\tilde{t}(X) \tag{1}$$

where $\bar{\theta}(X)$ denotes the dual of $\theta(X)$ and $\tilde{t}(X)$ denotes the 2-form corresponding to the skew-symmetric part of $t(X)$ [3].

LEMMA 1. *For any vector field X on a Kaehler manifold M*

$$\|\theta(X)\omega\|^2 = \|\delta\xi\|^2 + 2(\tilde{t}(X), \theta(X)\omega)$$

where $\|\alpha\|^2 = (\alpha, \alpha) = \int_M \alpha \wedge {}^*\alpha$ for any p-form α.

THEOREM 2. *A vector field X defines an infinitesimal analytic transformation of a Kaehler manifold if and only if $J\theta(X)\omega = \theta(X)g$, that is when applied to ω the operators $\theta(X)$ and J commute.*

This follows from the fact that $t(X) = \theta(X)\omega + J\theta(X)g$ a relation used in establishing formula (1). Lemma 1 therefore implies the following

COROLLARY. *For an infinitesimal analytic transformation*

$$\|\theta(X)\omega\| = \|\delta\xi\|.$$

Hence, a divergence free analytic map is an infinitesimal automorphism of the Kaehler structure.

LEMMA 2. *For an infinitesimal conformal transformation X of a Kaehler manifold*

$$t(X) = \theta(X)\omega + \frac{1}{n}\,\delta\xi \cdot \omega.$$

Our notation does not distinguish between the 2-form $t(X)$ and the corresponding tensorfield. If ξ is closed, $t(X)$ is symmetric and must therefore vanish. We conclude from the lemma that $d\delta\xi = 0$ for $n > 1$, that is X is homothetic. Since a homothetic map of a complete Riemannian manifold which is not locally flat is isometric we conclude

THEOREM 3. *A closed infinitesimal conformal transformation of a complete Kaehler manifold M^{2n} $(n > 1)$ which is not locally flat is an automorphism of the Kaehler structure.*

In the locally flat case an infinitesimal affine transformation X is isometric if and only if its length is bounded, that is the vector field on M defining X has bounded length. Hence, since a homothetic map is affine we have

THEOREM 4. *A closed infinitesimal conformal map of a complete locally flat Kaehler manifold* M^{2n} $(n>1)$ *is an automorphism of the Kaehler structure if and only if its length is bounded.*

REMARKS. (a) Every conformal map of a complete flat space is homothetic.

(b) M. Obata has communicated to us the following result (unpublished): "A closed infinitesimal conformal transformation of a (locally) reducible Riemannian manifold is homothetic." This means that only an irreducible Riemannian manifold can admit closed nonhomothetic maps.

3. **Proof of Theorem 1.** It is first shown that

$$\theta(X)\omega + \bar{\theta}(X)\omega = \left(1 - \frac{2}{n}\right)\delta\xi\cdot\omega. \tag{3}$$

Since $\bar{\theta}(X) = \epsilon(\xi)\delta + \delta\epsilon(\xi)$ (where $\epsilon(\xi)\alpha = \xi\wedge\alpha$ for any p-form α), δ and $\bar{\theta}(X)$ commute. Hence, applying δ to both sides of (3) we obtain

$$\delta\theta(X)\omega = -\left(1 - \frac{2}{n}\right)Cd\delta\xi.$$

Taking the global scalar product with $C\xi$ we derive

$$||\theta(X)\omega||^2 = -\left(1 - \frac{2}{n}\right)||\delta\xi||^2.$$

For $n>1$, $\theta(X)\omega$ vanishes. Moreover, $\delta\xi = 0$, that is the automorphisms are isometries.

4. Bochner and Montgomery [1] have shown that the group G of analytic homeomorphisms of a compact complex manifold M is a Lie group. If M is an Einstein Kaehler manifold G is reductive [6]. More generally, if the Ricci scalar curvature of a compact Kaehler manifold is a (positive) constant the same conclusion is valid [5]. This seems to be the best possible generalization of the result of [6] as one may see by considering the Gaussian 2-sphere with any metric with nonconstant scalar curvature. By restricting the analytic maps to those which are closed in the above sense no restrictions of a local nature regarding curvature are required and results parallel to Theorems 3 and 4 may be obtained.

REMARK. From the proof of Theorem 3 it follows that the image by J of a closed infinitesimal conformal transformation of a Kaehler manifold is an infinitesimal isometry. In fact

THEOREM 5. *A closed infinitesimal conformal transformation of a Kaehler manifold is an infinitesimal analytic transformation whose image by J is an infinitesimal isometry.*

For noncompact almost Kaehler manifolds we may prove

THEOREM 6. *If the largest connected group of automorphisms is a semi-simple Lie group its elements are volume preserving transformations.*

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