Consider the sequence of polynomials \( \{u_n(x)\} \) that satisfy the recurrence
\[
u_{n+1}(x) = (x + a(n))u_n(x) + b(n)u_{n-1}(x),
\]
where \( a(n), b(n) \) are polynomials in \( n \) (and possibly some additional indeterminates) with integral coefficients. Moreover it is assumed that
\[
u_0(x) = 1, \quad u_1(x) = a(0), \quad b(0) = 0.
\]
The sequence \( \{u_n(x)\} \) is uniquely determined by (1) and (2).

The writer [1, Theorem 1] has proved that if \( m^1, r^1 \), then \( u_n(x) \) satisfies the congruence
\[
\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x)u_{(r-s)m}(x) \equiv 0 \pmod{m^r},
\]
for all \( n \geq 1 \), where
\[
r = \lceil (r + 1)/2 \rceil,
\]
the greatest integer \( \leq (r+1)/2 \). In the present paper it is proved that \( u_n(x) \) satisfies the simpler congruence
\[
\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x)u_{m^s}(x) \equiv 0 \pmod{m^r},
\]
where again \( r_1 \) is defined by (4). Also it is shown that (5) implies
\[
\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x)u_{b_1(r-s)m}(x) \equiv 0 \pmod{m^r},
\]
for all \( n \geq 0, k \geq 0; \) for \( k = 0, \) (6) evidently reduces to (3). Indeed if we put
\[
U_k^{(r)} = U_{n_1, \ldots, n_k}(x) = \sum_{s_1+\cdots+s_k=r} \frac{r!}{s_1! \cdots s_k!} \lambda_1^{s_1} \cdots \lambda_k^{s_k} \prod_{j=1}^{k} u_{n_j + s_j m}(x),
\]
where \( \lambda_1, \ldots, \lambda_k \) are rational numbers that are integral (mod \( m \)) and such that
\[
\lambda_1 + \cdots + \lambda_k \equiv 0 \pmod{m},
\]
then it is shown that
\[
(7) \quad U_k^{(r)} \equiv 0 \pmod{m^r}
\]
for all \( n_1, \ldots, n_k \geq 0. \)

We remark that the congruence (7) was suggested by certain congruences for the Bernoulli numbers that were obtained by Van­
diver [2].

There are numerous applications of (5). In particular we mention the following which is related to elliptic functions. The Stieltjes formula [3, p. 374]
\[
\int_0^\infty sn(u, k^2)e^{-xu}du = \frac{1}{x^2 + a} - \frac{1 \cdot 2^2 \cdot 3k^2}{x^2 + 3^2 a} - \frac{3 \cdot 4^2 \cdot 5k^2}{x^2 + 5^2 a} - \cdots,
\]
where \( a = 1 + k^2, \) suggests the consideration of the polynomials \( f_n(x) \) defined by
\[
(8) \quad f_{n+1}(x) = (x + (2n + 1)^2a)f_n(x) - (2n - 1)(2n + 1)k^2 f_{n-1}(x),
\]
together with \( f_0(x) = 1, f_1(x) = x + a. \) Since (8) is of the form (1), it follows that these polynomials satisfy (5). Similar results hold for the polynomials associated in like manner with the integrals
\[
x \int_0^\infty sn(u, k^2)e^{-xu}du, \quad \int_0^\infty cn(u, k^2)e^{-xu}du, \quad \int_0^\infty dn(u, k^2)e^{-xu}du.
\]
We remark that (8) implies
\[
\sum_{n=0}^\infty f_n(x^2) \frac{sn^{2n+1} u}{(2n + 1)!} = \frac{\sinh xu}{x}.
\]

We show also that if \( p = 2w + 1 \) is an odd prime then \( f(x) \equiv \bar{f}(x) \pmod{p}, \) where
\[
(9) \quad \bar{f}_p(x) = x \{x^w - C_p(k^2)\}^2
\]
and
Thus (5) reduces to

\[ \sum_{s=0}^{r} (-1)^s \binom{r}{s} f_{n+sp}(x) f_{p}^{r-s}(x) \equiv 0 \pmod{p^r}, \]

where \( f_p(x) \) is defined by (9) and (10).

REFERENCES