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ARITHMETIC PROPERTIES OF CERTAIN POLYNOMIAL SEQUENCES

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Consider the sequence of polynomials $\{u_n(x)\}$ that satisfy the recurrence

$$u_{n+1}(x) = (x + a(n))u_n(x) + b(n)u_{n-1}(x), \tag{1}$$

where $a(n)$, $b(n)$ are polynomials in n (and possibly some additional indeterminates) with integral coefficients. Moreover it is assumed that

$$u_0(x) = 1, \qquad u_1(x) = a(0), \qquad b(0) = 0. \tag{2}$$

The sequence $\{u_n(x)\}$ is uniquely determined by (1) and (2).

The writer [1, Theorem 1] has proved that if $m \geqq 1$, $r \geqq 1$, then $u_n(x)$ satisfies the congruence

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x) u_{(r-s)m}(x) \equiv 0 \pmod{m^{r_1}}, \tag{3}$$

for all $n \geqq 1$, where

$$r_1 = [(r+1)/2], \tag{4}$$

the greatest integer $\leqq (r+1)/2$. In the present paper it is proved that $u_n(x)$ satisfies the simpler congruence

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x) u_m^{r-s}(x) \equiv 0 \pmod{m^{r_1}}, \tag{5}$$

where again r_1 is defined by (4). Also it is shown that (5) implies

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x) u_{k+(r-s)m}(x) \equiv 0 \pmod{m^{r_1}}, \tag{6}$$

for all $n \geqq 0$, $k \geqq 0$; for $k=0$, (6) evidently reduces to (3). Indeed if we put

$$U_k^{(r)} = U_{n_1,\cdots,n_k}^{(r)}(x) = \sum_{s_1+\cdots+s_k=r} \frac{r!}{s_1!\cdots s_k!} \lambda_1^{s_1} \cdots \lambda_k^{s_k} \prod_{j=1}^{k} u_{n_j+s_jm}(x),$$

where $\lambda_1, \cdots, \lambda_k$ are rational numbers that are integral (mod m) and such that

$$\lambda_1 + \cdots + \lambda_k \equiv 0 \pmod{m},$$

then it is shown that

$$U_k^{(r)} \equiv 0 \pmod{m^{r_1}} \tag{7}$$

for all $n_1, \cdots, n_k \geqq 0$.

We remark that the congruence (7) was suggested by certain congruences for the Bernoulli numbers that were obtained by Vandiver [**2**].

There are numerous applications of (5). In particular we mention the following which is related to elliptic functions. The Stieltjes formula [**3**, p. 374]

$$\int_0^\infty sn(u, k^2)e^{-xu}du = \frac{1}{x^2 + a-}\ \frac{1\cdot 2^2\cdot 3k^2}{x^2 + 3^2a-}\ \frac{3\cdot 4^2\cdot 5k^2}{x^2 + 5^2a-} \cdots,$$

where $a=1+k^2$, suggests the consideration of the polynomials $f_n(x)$ defined by

$$f_{n+1}(x) = (x + (2n+1)^2a)f_n(x) - (2n-1)(2n)^2(2n+1)k^2f_{n-1}(x), \tag{8}$$

together with $f_0(x)=1$, $f_1(x)=x+a$. Since (8) is of the form (1), it follows that these polynomials satisfy (5). Similar results hold for the polynomials associated in like manner with the integrals

$$x\int_0^\infty sn^2(u, k^2)e^{-xu}du, \qquad \int_0^\infty cn(u, k^2)e^{-xu}du, \qquad \int_0^\infty dn(u, k^2)e^{-xu}du.$$

We remark that (8) implies

$$\sum_{n=0}^\infty f_n(x^2) \frac{sn^{2n+1}u}{(2n+1)!} = \frac{\sinh xu}{x}.$$

We show also that if $p=2w+1$ is an odd prime then $f(x) \equiv \bar{f}(x)$ (mod p), where

$$\bar{f}_p(x) = x\{x^w - C_p(k^2)\}^2 \tag{9}$$

and

$$C_p(k^2) = (-1)^w \sum_{s=0}^{w} \binom{w}{s}^2 k^{2s}. \tag{10}$$

Thus (5) reduces to

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} f_{n+sp}(x) \bar{f}_p^{r-s}(x) \equiv 0 \pmod{p^{r_1}}, \tag{11}$$

where $\bar{f}_p(x)$ is defined by (9) and (10).

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