THE WEAK HAUPTVERMUTUNG FOR CELLS AND SPHERES

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THEOREM. *If P and Q are two triangulations of the n-sphere (closed n-cell), there is a third triangulation M which can be obtained from either by subdivision. In fact, M can be obtained from either P or Q by subdivision of a single n-simplex.*

The following result, obtained recently by M. Brown [1], is the principal tool of both proofs.

LEMMA. *Let S^{n-1} be an $n-1$ sphere embedded in the n-sphere S^n. If S^{n-1} has a neighborhood in S^n homeomorphic to $S^{n-1}\times[-1, 1]$, in which S^{n-1} is embedded as $S^{n-1}\times 0$, then the closures of the complementary domains of S^{n-1} in S^n are both closed n-cells.*

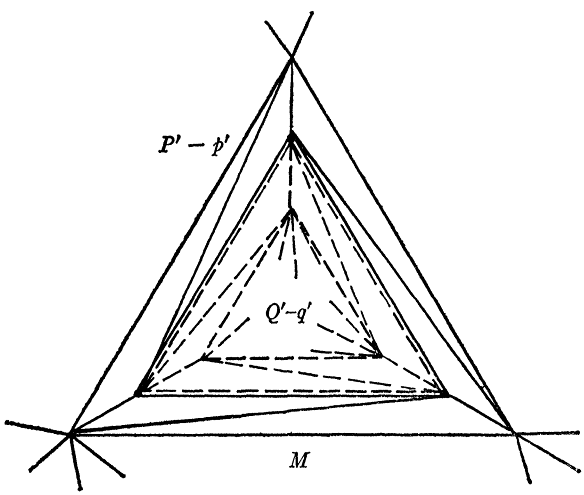

FIG. 1

We prove the theorem first for the n-sphere. Let p be an n-simplex of P, q an n-simplex of Q. Let p' be a smaller, concentric n-simplex inside p, and let P' be obtained from P by drawing p' inside p and triangulating the region ($S^{n-1}\times[0, 1]$) between the boundaries of p and p'. Similarly for q' and Q'. The boundaries of $|p'|$ and $|q'|$ have neighborhoods as required in the lemma, so they split $|P'|$, resp. $|Q'|$, into two closed n-cells, one of which is $|p'|$, resp. $|q'|$, and the other $|P'-p'|$, resp. $|Q'-q'|$. Let ω be a simplicial homeomorphism sending the boundary of p' onto the boundary of q'. The complex $M=(P'-p')\cup_\omega(Q'-q')$, which clearly triangulates an n-sphere, can be obtained from P by subdivision of p, and from Q by subdivision of q. Figure 1 illustrates the construction of M.

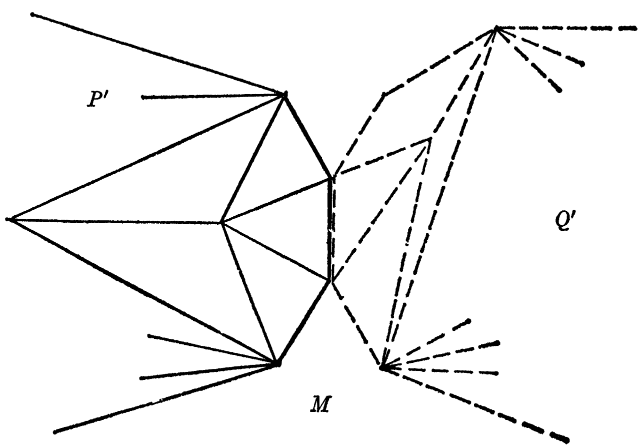

FIG. 2

The closed n-cell is treated similarly. Let p be an $n-1$ simplex on the boundary of $|P|$, and let α be the vertex opposite p in the n-simplex $\alpha\circ p$ of P containing p. Let p' be drawn inside p and concentric with it, and triangulate the region $(S^{n-2}\times[0, 1])$ between the boundaries of p and p'. Let α' be the barycenter of $\alpha\circ p$. Joining α' to the already subdivided boundary of $\alpha\circ p$ yields a subdivision P' of P. The n-simplex of P' containing p' as a face is $\alpha'\circ p'$. The boundary of $|p'|$ has a neighborhood on the boundary of $|P'|$ which satisfies the conditions of the lemma. It thus splits the boundary of the n-cell into two closed $n-1$ cells with a common boundary. Let the same steps be taken with the complex Q, leading to subdivision Q' containing an $n-1$ simplex q' on the boundary of $|Q'|$. q' will, in turn, be a face of the n simplex $\beta'\circ q'$ of Q'.

Let ω be a simplicial homeomorphism sending p' onto q'. Setting $M=P'\cup_\omega Q'$, we notice that an immediate consequence of the fact that the boundary of $|p'|$, resp. $|q'|$, splits the boundary of $|P'|$, resp. $|Q'|$, into two closed $n-1$ cells is that $|M|$ is a closed n-cell. Furthermore, $\alpha'\circ p'\cup_\omega Q'$ is isomorphic to a subdivision of $\alpha'\circ p'$, and $P'\cup_\omega\beta'\circ q'$ to a subdivision of $\beta'\circ q'$ by a similar argument. M can therefore be obtained from P by subdivision of $\alpha\circ p$ and from

Q by subdivision of $\beta \circ q$. Figure 2 exhibits the construction of M.

Our method does not demonstrate the combinatorial equivalence of P and Q, and is therefore only a verification of a weakened form of the full Hauptvermutung (see [2]). We observe, however, that the subdivision of P into M is as "nice" as is the complex Q, and the subdivision of Q into M is as "nice" as is the complex P. If, for example, Q is a combinatorial cell or sphere, then P is combinatorially equivalent to M. If both P and Q are combinatorial cells or spheres, we obtain a special case of a classical result due to Newman, [3].

References

1. M. Brown, *A proof of the generalized Schönflies theorem*, Notices Amer. Math. Soc. vol. 7 (1959) p. 783.

2. R. Furch, *Zur Grundlegung der kombinatorischen Topologie*, Abh. Math. Sem. Hansischen Univ. vol. 3 (1924) pp. 69–88.

3. M. H. A. Newman, *On the superposition of n-dimensional manifolds*, J. London Math. Soc. vol. 2 (1927) pp. 56–64.

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