A local ring is a commutative ring $L$ with unit which has a greatest ideal $I \neq L$. The set $L^* = L - I$ of units in a local ring $L$ forms a group under the multiplication. $L/I$ is a field, the so-called residue field of $L$. The homomorphic image of a local ring, if it is not the zero ring $0$, is again a local ring.

An $n$-dimensional vector space over $L$, $V_n(L)$, is a $L$-module isomorphic to $L^n$. An $m$-dimensional subspace $W$ of $V = V_n(L)$ is a direct summand isomorphic to $L^m$.

The general linear group in $n$ variables over $L$, $GL(n, L)$, is the group of linear automorphisms of $V_n(L)$. We propose to study the structure of this group, more precisely, we wish to describe the position of the invariant subgroups of $GL(n, L)$. In the case that $L$ is a field it is well known that $GL(n, L)$ has only big and small invariant subgroups, that is to say, in this case $GL(n, L)$ has only invariant subgroups which either contain the special linear group $SL(n, L)$ or else are contained in the center $Z(GL(n, L))$ of $GL(n, L)$, cf. Dieudonné [3] and [4] and Artin [1]. If $L$ is not a field, however, there will be non-trivial ideals in $L$ which give rise to more invariant subgroups, the so-called congruence subgroups modulo an ideal $J$ of $L$. Our main result is, cf. Theorem 3 below, that for a local ring $L$ it is still possible to get a survey over the different invariant subgroups $G$ of $GL(n, L)$, each of which is determining an ideal $J$ of $L$. Our main result is, cf. Theorem 3 below, that for a local ring $L$ it is still possible to get a survey over the different invariant subgroups $G$ of $GL(n, L)$, each of which is determining an ideal $J$ of $L$ such that $G$ is situated between a greatest and a smallest congruence subgroup mod $J$. In the case that this ideal $J$ is $L$ or $0$ (the only possibilities if $L$ is a field) these greatest and smallest congruence subgroups are $GL(n, L)$ and $SL(n, L)$ (for $J = L$) and $E = unit$ group (for $J = 0$) respectively.

Let $J$ be an ideal of the local ring $L$. Denote by $g_J$ the natural homomorphism $L \to L/J$. By the same letter we denote the natural homomorphism $g_J : V_n(L) \to V_n(L/J)$. $g_J$ determines the homomorphism

$$h_J : GL(n, L) \to GL(n, L/J)$$

with $h_J \sigma g_J = g_J \sigma$ for $\sigma \in GL(n, L)$.

Let $J$ be an ideal of $L$. The general congruence subgroup mod $J$ of $GL(n, L)$, $GC(n, L, J)$, is defined by
The center $Z(GL(n, L/J))$ consists of the homotheties and is hence isomorphic to the multiplicative group $(L/J)^*$. $GC(n, L, J)$ is an invariant subgroup of $GL(n, L)$. If $J = L$, $GL(n, L/J)$ denotes the unit group $E$. Note: $GC(n, L, L) = GL(n, L)$; $GC(n, L, 0) = Z(GL(n, L))$.

The order $o(X)$ of a vector $X \in V_n(L)$ is the smallest ideal $J$ with $g_JX = 0$. The order $o(\sigma)$ of an element $\sigma \in GL(n, L)$ is the smallest ideal $J$ with $h_J \sigma \in Z(GL(n, L/J))$. The order $o(G)$ of a subgroup $G$ of $GL(n, L)$ is the smallest ideal $J$ with $h_J G \subseteq Z(GL(n, L/J))$, i.e. $GC(n, L, J) \supseteq G$. Note: $o(X)$ and $o(\sigma)$ are finitely generated. $o(G)$ is generated by the ideals $o(\sigma)$ where $\sigma$ runs through $G$.

An element $\tau \in GL(n, L)$ is called a transvection, if there is a subspace $H$ in $V = V_n(L)$ of codimension one such that $\tau | H = \text{identity}$ and $\tau X - X \in H$ for all $X \in V$. Using a linear form $\phi$ with $H = \phi^{-1}(0)$, $\tau$ can be written as $\tau X = X + A\phi(X)$. We have $o(A) = o(\tau)$.

Now one proves the fundamental

LEMMA. Two vectors $A$ and $B$ of $V_n(L)$ have the same order if and only if there is an element $\sigma \in GL(n, L)$ which carries $A$ into $B$.

From this one deduces in the usual way

PROPOSITION 1. Any two transvections of the same order are conjugate under $GL(n, L)$.

Let $J$ be an ideal of $L$. The special congruence subgroup mod $J$ of $GL(n, L)$, $SC(n, L, J)$, is defined as the invariant subgroup generated by the transvections of order $\leq J$. Note: $SC(n, L, 0) = E$; $SC(n, L, L) = SL(n, L)$, the special linear group in $n$ variables over $L$.

The following theorem generalizes well known characterizations of the special linear group over a field:

THEOREM 1. Let $G$ be a subgroup of $GL(n, L)$, $J$ an ideal of $L$. The following statements are equivalent:

(a) $G = SC(n, L, J)$.

(b) $G = \text{the group formed by the elements } \sigma \in GL(n, L) \text{ with } \det \sigma = 1 \text{ and } h_J \sigma = \text{identity}$.

(c) $G = \text{the mixed commutator group } K(GL(n, L), GC(n, L, J))$. Here we assume for $n = 2$ that $L/I \not\cong F_2$.

A simple consequence is the

THEOREM 2. $GC(n, L, J)/SC(n, L, J)$ is commutative. More precisely, it is isomorphic to the subgroup of $L^* \times (L/J)^*$ formed by the pairs $(a, b)$ with $g_J a = b^n$. 

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Consequently, any subgroup $G$ of $GL(n, L)$ with $GC(n, L, J) \supset G \supset SC(n, L, J)$ is an invariant subgroup of $GL(n, L)$ of order $o(G) = J$.

It remains to be seen whether any invariant subgroup $G$ of order $J$ lies between $GC(n, L, J)$ and $SC(n, L, J)$. First one proves

**Proposition 2.** Let $\tau$ be a transvection of order $J$. Then the invariant subgroup in $GL(n, L)$, generated by $\tau$, is $SL(n, L, J)$. Here we assume for $n = 2$: char $(L/I) \neq 2$.

**Proposition 3.** Let $G$ be a subgroup of $GL(n, L)$, invariant under $SL(n, L)$. Then $G$ contains, for each $\sigma \in G$, the transvections of order $<\sigma$. Here we assume for $n = 2$: char $(L/I) \neq 2$ and $L/I \neq F_3$.

Combining these results we get the following

**Theorem 3.** An invariant subgroup $G$ of $GL(n, L)$ determines an ideal $J$ of $L$ such that

\[ (*) \quad GC(n, L, J) \supset G \supset SC(n, L, J). \]

Conversely, any subgroup $G$ of $GL(n, L)$ which satisfies $(*)$ is invariant and of order $o(G) = J$. The invariant subgroups of $GL(n, L)$ are therefore in one to one correspondence with the pairs consisting of an ideal $J$ of $L$ and a subgroup $GC(n, L, J)/SC(n, L, J)$. Here we assume for $n = 2$: char $(L/I) \neq 2$ and $L/I \neq F_3$.

**Remarks**

1. The preceding theorem contains, as a special case, the well known structure theorem of the general linear group over a commutative field, cf. Dieudonné [3; 4] and Artin [1].

2. One easily deduces from the preceding theorem a structure theorem for the special linear group over a local ring.

3. If $L$ is especially the ring of the integers modulo the power of a prime, the preceding theorem has been proved by Brenner [2].

**Added in proof:** Since the completion of this note, similar results have been obtained for the orthogonal groups over local rings.

**References**


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