LINEAR GROUPS OVER LOCAL RINGS

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A *local ring* is a commutative ring L with unit which has a greatest ideal $I \neq L$. The set $L^* = L - I$ of units in a local ring L forms a group under the multiplication. L/I is a field, the so-called residue field of L. The homomorphic image of a local ring, if it is not the zero ring 0, is again a local ring.

An *n-dimensional vector space over* L, $V_n(L)$, is a L-module isomorphic to L^n. An *m-dimensional subspace* W *of* $V = V_n(L)$ is a direct summand isomorphic to L^m.

The *general linear group in n variables over* L, $GL(n, L)$, is the group of linear automorphisms of $V_n(L)$. We propose to study the structure of this group, more precisely, we wish to describe the position of the invariant subgroups of $GL(n, L)$. In the case that L is a field it is well known that $GL(n, L)$ has only big and small invariant subgroups, that is to say, in this case $GL(n, L)$ has only invariant subgroups which either contain the special linear group $SL(n, L)$ or else are contained in the center $Z(GL(n, L))$ of $GL(n, L)$, cf. Dieudonné [3] and [4] and Artin [1]. If L is not a field, however, there will be non-trivial ideals in L which give rise to more invariant subgroups, the so-called congruence subgroups modulo an ideal J of L. Our main result is, cf. Theorem 3 below, that for a local ring L it is still possible to get a survey over the different invariant subgroups G of $GL(n, L)$, each of which is determining an ideal J of L such that G is situated between a greatest and a smallest congruence subgroup mod J. In the case that this ideal J is L or 0 (the only possibilities if L is a field) these greatest and smallest congruence subgroups are $GL(n, L)$ and $SL(n, L)$ (for $J = L$) and $Z(GL(n, L))$ and E = unit group (for $J = 0$) respectively.

Let J be an ideal of the local ring L. Denote by g_J the natural homomorphism $L \to L/J$. By the same letter we denote the natural homomorphism g_J: $V_n(L) \to V_n(L/J)$. g_J determines the homomorphism

$$h_J: GL(n, L) \to GL(n, L/J)$$

with $h_J \sigma g_J = g_J \sigma$ for $\sigma \in GL(n, L)$.

Let J be an ideal of L. The *general congruence subgroup* mod J *of* $GL(n, L)$, $GC(n, L, J)$, is defined by

$$GC(n, L, J) = h_J^{-1} Z(GL(n, L/J)).$$

The center $Z(GL(n, L/J))$ consists of the homotheties and is hence isomorphic to the multiplicative group $(L/J)^*$. $GC(n, L, J)$ is an invariant subgroup of $GL(n, L)$. If $J=L$, $GL(n, L/J)$ denotes the unit group E. Note: $GC(n, L, L)=GL(n, L)$; $GC(n, L, 0)=Z(GL(n, L))$.

The *order* $o(X)$ *of a vector* $X \in V_n(L)$ is the smallest ideal J with $g_J X=0$. The *order* $o(\sigma)$ *of an element* $\sigma \in GL(n, L)$ is the smallest ideal J with $h_J\sigma \in Z(GL(n, L/J))$. The *order* $o(G)$ *of a subgroup* G *of* $GL(n, L)$ is the smallest ideal J with $h_J G \subset Z(GL(n, L/J))$, i.e. $GC(n, L, J) \supset G$. Note: $o(X)$ and $o(\sigma)$ are finitely generated. $o(G)$ is generated by the ideals $o(\sigma)$ where σ runs through G.

An element $\tau \in GL(n, L)$ is called a *transvection*, if there is a subspace H in $V=V_n(L)$ of codimension one such that $\tau | H=$identity and $\tau X - X \in H$ for all $X \in V$. Using a linear form ϕ with $H=\phi^{-1}(0)$, τ can be written as $\tau X = X + A\phi(X)$. We have $o(A)=o(\tau)$.

Now one proves the fundamental

LEMMA. *Two vectors* A *and* B *of* $V_n(L)$ *have the same order if and only if there is an element* $\sigma \in GL(n, L)$ *which carries* A *into* B.

From this one deduces in the usual way

PROPOSITION 1. *Any two transvections of the same order are conjugate under* $GL(n, L)$.

Let J be an ideal of L. The *special congruence subgroup* mod J *of* $GL(n, L)$, $SC(n, L, J)$, is defined as the invariant subgroup generated by the transvections of order $\subset J$. Note: $SC(n, L, 0)=E$; $SC(n, L, L) = SL(n, L)$, *the special linear group in* n *variables over* L.

The following theorem generalizes well known characterizations of the special linear group over a field:

THEOREM 1. *Let* G *be a subgroup of* $GL(n, L)$, J *an ideal of* L. *The following statements are equivalent*:

(a) $G=SC(n, L, J)$.

(b) $G=$ *the group formed by the elements* $\sigma \in GL(n, L)$ *with* $\det \sigma = 1$ *and* $h_J\sigma=$ *identity*.

(c) $G=$ *the mixed commutator group* $K(GL(n, L), GC(n, L, J))$. *Here we assume for* $n=2$ *that* $L/I \neq F_2$.

A simple consequence is the

THEOREM 2. $GC(n, L, J)/SC(n, L, J)$ *is commutative. More precisely, it is isomorphic to the subgroup of* $L^* \times (L/J)^*$ *formed by the pairs* (a, b) *with* $g_J a = b^n$.

Consequently, any subgroup G of $GL(n, L)$ with $GC(n, L, J) \supset G \supset SC(n, L, J)$ is an invariant subgroup of $GL(n, L)$ of order $o(G) = J$. It remains to be seen whether any invariant subgroup G of order J lies between $GC(n, L, J)$ and $SC(n, L, J)$. First one proves

PROPOSITION 2. *Let τ be a transvection of order J. Then the invariant subgroup in $GL(n, L)$, generated by τ, is $SL(n, L, J)$. Here we assume for $n=2$:* char $(L/I) \neq 2$.

PROPOSITION 3. *Let G be a subgroup of $GL(n, L)$, invariant under $SL(n, L)$. Then G contains, for each $\sigma \in G$, the transvections of order $\subset o(\sigma)$. Here we assume for $n=2$:* char $(L/I) \neq 2$ *and $L/I \neq F_3$.*

Combining these results we get the following

THEOREM 3. *An invariant subgroup G of $GL(n, L)$ determines an ideal J of L such that*

$$(*) \qquad GC(n, L, J) \supset G \supset SC(n, L, J).$$

Conversely, any subgroup G of $GL(n, L)$ which satisfies $(*)$ is invariant and of order $o(G) = J$. The invariant subgroups of $GL(n, L)$ are therefore in one to one correspondence with the pairs consisting of an ideal J of L and a subgroup $GC(n, L, J)/SC(n, L, J)$. Here we assume for $n=2$: char$(L/I) \neq 2$ *and $L/I \neq F_3$.*

REMARKS 1. The preceding theorem contains, as a special case, the well known structure theorem of the general linear group over a commutative field, cf. Dieudonné [**3**; **4**] and Artin [**1**].

2. One easily deduces from the preceding theorem a structure theorem for the special linear group over a local ring.

3. If L is especially the ring of the integers modulo the power of a prime, the preceding theorem has been proved by Brenner [**2**].

Added in proof: Since the completion of this note, similar results have been obtained for the orthogonal groups over local rings.

REFERENCES

1. E. Artin, *Geometric algebra*, New York, 1957.
2. J. Brenner, *The linear homogeneous group*, Ann. of Math. vol. 39 (1938) pp. 472–493; *The linear homogeneous group* II, Ann. of Math. vol. 45 (1944) pp. 100–109.
3. J. Dieudonné, *Sur les groupes classiques*, Paris, 1948.
4. ———, *La géométrie des groupes classiques*, Berlin, 1955.

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