(B) $$_R\sum_X S(X, R) \subset (S_0, \epsilon R)S(X, R)$$

and $\sum$ *is an m-plane through P such that*

(C) $$(\sum, \epsilon R_0) \supset S_0.$$

Then if $\epsilon \leqq 2^{-2000N^2}$ *there will exist a topological m-disk* $\bar{S}$ *such that*

$$S_0 S\left(P, \frac{1}{16} R_0\right) \subset \bar{S} \subset S_0 S(P, R_0).$$

Where $S(x, r)$ is a solid ball of centre x and radius r while (y, δ) is the set of points lying within δ of the set y.

Oregon State College and
University of Bristol, England

A CHARACTERIZATION OF THE ALGEBRA OF ALL CONTINUOUS FUNCTIONS ON A COMPACT HAUSDORFF SPACE

BY YITZHAK KATZNELSON

Communicated by Walter Rudin, March 23, 1960

This note is a complement to [1]. We consider a commutative, semi-simple and self-adjoint Banach algebra B and assume that B has a unit element and is regular. By $\mathfrak{M}$ we denote the space of maximal ideals of B and, applying the Gelfand representation, we consider B as an algebra of continuous functions defined on $\mathfrak{M}$. It is obvious that if B is $C(\mathfrak{M})$ (the algebra of all the continuous functions on $\mathfrak{M}$) the idempotents in any quotient algebra of B are always bounded. We prove here that this property characterizes $C(\mathfrak{M})$ and give an application of this result.

Lemma 1. *Suppose that there exist constants* K *and* K_1, $K_1 < 1$ *such that to any real, (resp. non-negative) function* $f \in C(\mathfrak{M})$ *there exists an element* $f_1 \in B$ *such that* $\|f_1\| \leqq K \operatorname{Sup}_{M\in\mathfrak{M}} |f(M)|$, $f - f_1$ *is real (non-negative) and*

$$\operatorname{Sup}_{M\in\mathfrak{M}} |f(M) - f_1(M)| < K_1 \operatorname{Sup}_{M\in\mathfrak{M}} |f(M)| ;$$

then $B = C(\mathfrak{M})$ *and for any* $f \in B$ $\|f\| \leqq 4K(1-K_1)^{-1} \operatorname{Sup}_{M\in\mathfrak{M}} |f(M)|$.

Proof. Define by induction $f_n = (f - \sum^{n-1} f_i)_1$; then $f = \sum_1^\infty f_n$.

LEMMA 2. *Suppose that there exists a constant K_2 such that if h is an idempotent in any quotient algebra of B, $\|h\| < K_2$; then $B = C(\mathfrak{M})$.*

PROOF. The condition imposed in the statement of the lemma means that given two disjoint closed sets in $\mathfrak{M}$, there is an element $h \in B$ such that $h(M)$ is 1 on one set, 0 on the other set and $\|h\| \leqq K_2$. We may also assume that h is non-negative since we may replace it by $|h|^2$, taking, if necessary, a bigger K_2.

Let f be a non-negative function in $C(\mathfrak{M})$, define:

$$P_1 = \left\{M; f(M) \geqq \left(1 - \frac{1}{3K_2}\right) \operatorname{Sup} f\right\},$$

$$P_2 = \{M; f(M) \leqq 1/2 \operatorname{Sup} f\},$$

and let $h(M)$ be a non-negative element in B, of norm $\leqq K_2$ which is identically 1 on P_1 and vanishes on P_2. $f_1(M) = (2K_2)^{-1} \operatorname{Sup} f \cdot h(M)$ has the following properties: $\|f_1\| \leqq 1/2 \operatorname{Sup} f$, $f - f_1$ is non-negative and $\operatorname{Sup} (f - f_2) < (1 - (1/3K_2)) \operatorname{Sup} f$ and the lemma follows from Lemma 1 with $K = 1/2$ and $K_1 = 1 - (1/3K_2)$.

DEFINITION 1. $B(P)$, where P is closed in $\mathfrak{M}$, is the algebra of restrictions of B to P or, equivalently, the quotient algebra of B by the kernel of P.

DEFINITION 2. We say that B is bounded in a set $V \subseteq \mathfrak{M}$ if there exists a constant $K = K(V)$ such that whenever h is an idempotent in $B(P)$ with $P \subseteq V$, $\|h\| < K(V)$.

LEMMA 3. *Let B be bounded in V_1 and in V_2 where V_1 and V_2 are open in $\mathfrak{M}$. Then B is bounded in every closed subset of $V_1 \cup V_2$.*

PROOF. Let W be a closed subset of $V_1 \cup V_2$. We may assume $W = \mathfrak{M}$ (since we can confine our attention to $B(W)$ instead of B). There exist open sets W_1, W_2 satisfying: $\overline{W}_j \subset V_j$; $W_1 \cup W_2 = \mathfrak{M}$. Since B is regular it contains a function ϕ,

$$\phi(M) = \begin{cases} 0 & M \notin W_1, \\ 1 & M \notin W_2. \end{cases}$$

If P is closed in $\mathfrak{M}$, $P = (P \cap \overline{W}_1) \cup (P \cap \overline{W}_2)$ and every idempotent in $B(P)$ can be obtained as $\phi h_1 + (1 - \phi) h_2$ where h_j is an idempotent in $B(P \cap \overline{W}_j)$ and the lemma follows.

DEFINITION 3. B is bounded at a maximal ideal M if there is a neighborhood V of M such that B is bounded in V.

LEMMA 4. *Let P be compact in $\mathfrak{M}$; if B is bounded at every $M \in P$, there exists an open $V \supseteq P$ such that B is bounded in V.*

This is an obvious consequence of Lemma 3.

LEMMA 5. *If the idempotents of any quotient algebra of B are bounded, there is at most a finite number of points in* $\mathfrak{M}$ *at which B is not bounded.*

PROOF. If there were infinitely many there would exist a sequence $\{M_j\}_{j=1}^{\infty}$ with disjoint neighborhoods V_j such that B would not be bounded in V_j. There would be a closed $P_j \leqq V_j$ such that $B(P_j)$ would contain an idempotent of norm $\geqq j$.

If $P = \bigcup P_j$ then $B(\overline{P})$ would not have its idempotents bounded.

The preceding proof yields actually more. We see that under the conditions of Lemma 5, there exists, for every family of disjoint open sets $\{V_\alpha\}$, a constant K such that $K(V_\alpha) \leqq K$ for all but a finite number of α's.

Let us now show that, under the condition of Lemma 5, B is bounded at every $M \in \mathfrak{M}$. We may obviously assume that there is only one point "in doubt" and denote it by M_0.

There is a neighborhood V of M_0 and a constant K such that every idempotent in $B(P)$, where $P \subset V$ and has M_0 as an isolated point, has norm less than K. (Use the same argument as in the proof of Lemma 5.) Using Lemmas 3, 4 and the assumption that M_0 was the only point at which B was not known to be bounded we see that we may take $V = \mathfrak{M}$. For every closed P that has M_0 as an isolated point $B(P) = C(P)$ (Lemma 2) and there is a constant A, independent of P, such that the norm in $B(P)$ is bounded by A times the Sup norm. This implies [1, the last lemma] $B = C(\mathfrak{M})$ and we thus proved

THEOREM. *If the idempotents of any quotient algebra of B are bounded,* $B = C(\mathfrak{M})$.

COROLLARY (*For Terminology see* [1]). *If there is a function* $F(x)$ *defined for* $-1 < x < 1$ *that operates in B and such that*

$$F(0) = 0, \qquad \lim_{x \to 0} x^{-1}F(x) = \infty.$$

Then $B = C(\mathfrak{M})$.

PROOF. Use [1, Theorem 1] and the fact that F operates also in any quotient algebra of B.

REFERENCE

1. Y. Katznelson, *Sur les algèbras dont les éléments non-négatifs admettent des racines carrées*, Ann. Sci. Ecole Norm. Sup. vol. 77 (1960).

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIFORNIA