A CHARACTERIZATION OF THE ALGEBRA OF ALL CONTINUOUS FUNCTIONS ON A COMPACT HAUSDORFF SPACE

BY YITZHAK KATZNELSON

Communicated by Walter Rudin, March 23, 1960

This note is a complement to [1]. We consider a commutative, semi-simple and self-adjoint Banach algebra $B$ and assume that $B$ has a unit element and is regular. By $\mathcal{M}$ we denote the space of maximal ideals of $B$ and, applying the Gelfand representation, we consider $B$ as an algebra of continuous functions defined on $\mathcal{M}$. It is obvious that if $B$ is $C(\mathcal{M})$ (the algebra of all the continuous functions on $\mathcal{M}$) the idempotents in any quotient algebra of $B$ are always bounded. We prove here that this property characterizes $C(\mathcal{M})$ and give an application of this result.

**Lemma 1.** Suppose that there exist constants $K$ and $K_1$, $K_1<1$ such that to any real, (resp, non-negative) function $f \in C(\mathcal{M})$ there exists an element $f_i \in B$ such that $\|f_i\| \leq K \sup_{M \in \mathcal{M}} |f(M)|$, $f - f_i$ is real (non-negative) and

$$\sup_{M \in \mathcal{M}} |f(M) - f_i(M)| < K_1 \sup_{M \in \mathcal{M}} |f(M)|;$$

then $B = C(\mathcal{M})$ and for any $f \in B$ $\|f\| \leq 4K(1-K_1)^{-1} \sup_{M \in \mathcal{M}} |f(M)|$.

**Proof.** Define by induction $f_n = (f - \sum_{i=1}^{n-1} f_i)_i$; then $f = \sum_i f_n$. 

---

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Lemma 2. Suppose that there exists a constant $K_2$ such that if $h$ is an idempotent in any quotient algebra of $B$, $\|h\| < K_2$; then $B = C(\mathcal{W})$.

Proof. The condition imposed in the statement of the lemma means that given two disjoint closed sets in $\mathcal{W}$, there is an element $h \in B$ such that $h(M)$ is 1 on one set, 0 on the other set and $\|h\|\leq K_2$. We may also assume that $h$ is non-negative since we may replace it by $|h|^2$, taking, if necessary, a bigger $K_2$.

Let $f$ be a non-negative function in $C(\mathcal{W})$, define:

$$P_1 = \{ M; f(M) \geq \left(1 - \frac{1}{3K_2}\right) \text{Sup} f \},$$
$$P_2 = \{ M; f(M) \leq \frac{1}{2} \text{Sup} f \},$$

and let $h(M)$ be a non-negative element in $B$, of norm $\leq K_2$ which is identically 1 on $P_1$ and vanishes on $P_2$. $f_1(M) = (2K_2)^{-1} \text{Sup} f \cdot h(M)$ has the following properties: $\|f_1\| \leq 1/2 \text{Sup} f$, $f - f_1$ is non-negative and $\text{Sup} (f - f_2) < (1 - (1/3K_2)) \text{Sup} f$ and the lemma follows from Lemma 1 with $K = 1/2$ and $K_1 = 1 - (1/3K_2)$.

Definition 1. $B(P)$, where $P$ is closed in $\mathcal{W}$, is the algebra of restrictions of $B$ to $P$ or, equivalently, the quotient algebra of $B$ by the kernel of $P$.

Definition 2. We say that $B$ is bounded in a set $V \subseteq \mathcal{W}$ if there exists a constant $K = K(V)$ such that whenever $h$ is an idempotent in $B(P)$ with $P \subseteq V$, $\|h\| < K(V)$.

Lemma 3. Let $B$ be bounded in $V_1$ and in $V_2$ where $V_1$ and $V_2$ are open in $\mathcal{W}$. Then $B$ is bounded in every closed subset of $V_1 \cup V_2$.

Proof. Let $W$ be a closed subset of $V_1 \cup V_2$. We may assume $W = \mathcal{W}$ (since we can confine our attention to $B(W)$ instead of $B$). There exist open sets $W_1$, $W_2$ satisfying: $W_j \subseteq V_j$; $W_1 \cup W_2 = \mathcal{W}$. Since $B$ is regular it contains a function $\phi$,

$$\phi(M) = \begin{cases} 0 & M \in W_1, \\ 1 & M \in W_2. \end{cases}$$

If $P$ is closed in $\mathcal{W}$, $P = (P \cap \overline{W}_1) \cup (P \cap \overline{W}_2)$ and every idempotent in $B(P)$ can be obtained as $\phi h_1 + (1 - \phi) h_2$ where $h_i$ is an idempotent in $B(P \cap \overline{W}_i)$ and the lemma follows.

Definition 3. $B$ is bounded at a maximal ideal $M$ if there is a neighborhood $V$ of $M$ such that $B$ is bounded in $V$.

Lemma 4. Let $P$ be compact in $\mathcal{W}$; if $B$ is bounded at every $M \in P$, there exists an open $V \supseteq P$ such that $B$ is bounded in $V$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
This is an obvious consequence of Lemma 3.

**Lemma 5.** If the idempotents of any quotient algebra of $B$ are bounded, there is at most a finite number of points in $\mathbb{M}$ at which $B$ is not bounded.

**Proof.** If there were infinitely many there would exist a sequence $\{M_j\}_{j=1}^\infty$ with disjoint neighborhoods $V_j$ such that $B$ would not be bounded in $V_j$. There would be a closed $P_j \subseteq V_j$ such that $B(P_j)$ would contain an idempotent of norm $\geq j$.

If $P = \bigcup P_j$ then $B(\overline{P})$ would not have its idempotents bounded.

The preceding proof yields actually more. We see that under the conditions of Lemma 5, there exists, for every family of disjoint open sets $\{V_\alpha\}$, a constant $K$ such that $K(V_\alpha) \leq K$ for all but a finite number of $\alpha$'s.

Let us now show that, under the condition of Lemma 5, $B$ is bounded at every $M \subseteq \mathbb{M}$. We may obviously assume that there is only one point "in doubt" and denote it by $M_0$.

There is a neighborhood $V$ of $M_0$ and a constant $K$ such that every idempotent in $B(P)$, where $P \subseteq V$ and has $M_0$ as an isolated point, has norm less than $K$. (Use the same argument as in the proof of Lemma 5.) Using Lemmas 3, 4 and the assumption that $M_0$ was the only point at which $B$ was not known to be bounded we see that we may take $V = \mathbb{M}$. For every closed $P$ that has $M_0$ as an isolated point $B(P) = C(P)$ (Lemma 2) and there is a constant $A$, independent of $P$, such that the norm in $B(P)$ is bounded by $A$ times the Sup norm. This implies [1, the last lemma] $B = C(\mathbb{M})$ and we thus proved

**Theorem.** If the idempotents of any quotient algebra of $B$ are bounded, $B = C(\mathbb{M})$.

**Corollary** (For Terminology see [1]). If there is a function $F(x)$ defined for $-1 < x < 1$ that operates in $B$ and such that

$$F(0) = 0, \quad \lim_{x \to 0} x^{-1}F(x) = \infty.$$  

Then $B = C(\mathbb{M})$.

**Proof.** Use [1, Theorem 1] and the fact that $F$ operates also in any quotient algebra of $B$.

**Reference**


University of California,
Berkeley, California