# ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF BLOCK TOEPLITZ MATRICES

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Let $g(\lambda)$, $-\pi \leqq \lambda \leqq \pi$, be a $p \times p$ ($p=1, 2, \cdots$) matrix-valued Hermitian function. Further $g(\lambda)$ is bounded on $[-\pi, \pi]$, that is, its elements are bounded on $[-\pi, \pi]$. The Fourier coefficients

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\lambda} g(\lambda) d\lambda, \qquad k = 0, \pm 1, \cdots, \tag{1}$$

are then bounded in $k$. We call the $np \times np$ matrix

$$A_n = (a_{j-k}; j, k = 1, \cdots, n)$$

(an $n \times n$ matrix of the $p \times p$ blocks $a_{j-k}$) the $n$th section block Toeplitz matrix generated by $g(\lambda)$. Notice that the block Toeplitz matrix $A_n$ is generally not Toeplitz. Our interest is in obtaining the asymptotic distribution of eigenvalues of $A_n$ as $n \to \infty$. The proof is suggested by an argument given in the one-dimensional case ($p=1$) (see [3]) and is based on results in the multidimensional prediction problem [5].

If the real number $\alpha$ is sufficiently small in absolute value $f(\lambda) = [I_p + \alpha g(\lambda)]$ is positive definite for all $\lambda$ and bounded ($I_p$ is the identity matrix of order $p$). Let $R_n = I_{np} + \alpha A_n$ be the $n$th section block Toeplitz matrix generated by $f(\lambda)$. Further denote the $(i, j)$th block element ($p \times p$ matrix), $i, j = 1, \cdots, n$, of the inverse $R_n^{-1}$ of $R_n$ by ${}_n r_{i,j}^{(-1)}$. The basic result on the determinant of the prediction error covariance matrix in the multidimensional prediction problem [5] tells us that

$$\lim_{n\to\infty} \det \left({}_n r_{11}^{(-1)}\right)^{-1} = (2\pi)^p \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det \left( \frac{f(\lambda)}{2\pi} \right) d\lambda \right\}$$

since $f(\lambda)/2\pi$ can be regarded as the spectral density function of a $p$-vector weakly stationary stochastic process. However,

$$\det \left({}_n r_{11}^{(-1)}\right)^{-1} = \det (R_n)/\det (R_{n-1}) = \sigma_n^2$$

(see [1, p. 21]). Let $\lambda_{\nu,n}$, $\nu = 1, \cdots, np$, be the eigenvalues of $A_n$.

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Now

$$\det (R_n) = \prod_{\nu=1}^{np} (1 + \alpha\lambda_{\nu,n})$$

and

$$\lim \log \sigma_n^2 = \lim \frac{1}{n} \sum_{k=1}^{n} \log \sigma_k^2 = \lim \frac{1}{n} \sum_{\nu=1}^{np} \log (1 + \alpha\lambda_{\nu,n}).$$

Thus

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{np} \log (1 + \alpha\lambda_{\nu,n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det (I_p + \alpha g(\lambda)) d\lambda.$$

On taking $s_{n,k} = \sum_{\nu=1}^{np} \lambda_{\nu,n}^k$, it is readily seen that

$$\lim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{s_{n,k}}{pn} \alpha^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{1}{2\pi p} \int_{-\pi}^{\pi} \operatorname{tr}(g^k(\lambda)) \alpha^k d\lambda$$

for $|\alpha|$ sufficiently small ($\operatorname{tr}(A)$ denotes the trace of $A$). This cannot hold unless

$$\lim_{n\to\infty} \frac{s_{n,k}}{pn} = \frac{1}{2\pi p} \int_{-\pi}^{\pi} \operatorname{tr}(g^k(\lambda)) d\lambda. \tag{2}$$

*Let* $\mu_1(\lambda), \cdots, \mu_p(\lambda)$ *be the eigenvalues of* $g$ *at* $\lambda$, let us say for convenience in order of magnitude. Then relation (2) implies *that the asymptotic distribution of eigenvalues* $\lambda_{\nu,n} = 1, \cdots, pn$, *is given by*

$$\lim_{n\to\infty} \frac{\text{number of eigenvalues} \leqq a}{pn} = \frac{1}{p} \sum_{j=1}^{p} P[\mu_j(X) \leqq a] \tag{3}$$

*where* $X$ *is a random variable uniformly distributed on* $[-\pi, \pi]$. Aside from the interest in this result for its own sake, it is clear that it suggests a number of good approximations for the joint distribution of spectral estimates in the case of multidimensional normal stationary processes [2; 4].

## References


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