HOMOTOPY-ABELIAN LIE GROUPS

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A topological group G is said to be *homotopy-abelian* if the commutator map of $G\times G$ into G is nulhomotopic. Examples can be given[2] of non-compact Lie groups which are homotopy-abelian but not abelian. The purpose of this note is to prove

THEOREM. *A compact connected Lie group is homotopy-abelian only if it is abelian.*

COROLLARY. *If a Lie group is homotopy-abelian, then its maximal compact connected subgroup is abelian.*

Our proof depends on the theory of [6]. Thus we consider the Samelson "commutator" product[3] in the homotopy groups of G, which is trivial when G is homotopy-abelian. The product of $\alpha\in\pi_p(G)$ with $\beta\in\pi_q(G)$ is denoted by $\langle\alpha, \beta\rangle\in\pi_{p+q}(G)$, where $p, q\geqq 1$. If h is a homomorphism of G into another topological group then

$$h_*\langle\alpha, \beta\rangle = \langle h_*\alpha, h_*\beta\rangle,$$

where h_* denotes the induced homomorphism. Note that h_* is an isomorphism if h is a covering map and $p, q\geqq 2$. Hence if two topological groups have a common universal covering group then their higher homotopy groups are related by an isomorphism which is compatible with the Samelson product. Let $\sigma\,\pi_q(G)$, where $q\geqq 1$, denote the subset of $\pi_{2q}(G)$ consisting of elements $\langle\beta, \beta\rangle$, where $\beta\in\pi_q(G)$. We assert the following

LEMMA. *Let G be a compact connected simple non-abelian Lie group of dimension n and rank l. Then $\sigma\,\pi_q(G)\neq 0$, where $q=2n/l-3$.*

The proof is by application of (2.2) of [6]. We distinguish between the classical and exceptional cases, beginning with the latter.

Let G be one of the exceptional groups. Then $n/l=p$, an odd prime number, and G has no p-torsion (see [3]). The mod p cohomology of G is an exterior algebra on a basis of l generators. There is one generator y in dimension q, while the remainder are of lower dimension. It follows from Proposition 6 on page 291 of [8] that y has a nontrivial

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[2] Such as the 2-dimensional affine group (example suggested by H. Samelson).

[3] The theory of the Samelson product is given in [5], for example.

image under the homomorphism induced by some map of S^q into G. Thus y has nonzero index, in the sense of [6], with regard to some element $\beta \in \pi_q(G)$. By Borel's theorem[4] the mod p cohomology of B, the classifying space of G, is a polynomial algebra on a basis of l generators which correspond under transgression to those of the exterior algebra. The generator x corresponding to y has a nontrivial image under the homomorphism induced by some map of S^{q+1} into B. In the polynomial algebra let M denote the ideal generated by all the basis elements except x. If z is such a generator then

$$\dim z < \dim x = q + 1 = 2(p - 1),$$

and so $\mathcal{P}^s z \in M$, where $\mathcal{P}^s$ $(s \geqq 0)$ denotes the Steenrod operator. Hence $\mathcal{P}^s M \subset M$, by the Cartan product formula. This proves that $\mathcal{P}^1 x \notin M$, since by the Adem relation [1] we have

$$\mathcal{P}^{p-2}\mathcal{P}^1 x = (p - 1)\mathcal{P}^{p-1}x = (p - 1)x^p \notin M.$$

Hence $\mathcal{P}^1 x \equiv cx^2$, mod M, where $c \neq 0$, and so $\mathcal{P}^1 x$ is significant with regard to β, in the sense of [6]. Therefore $\langle \beta, \beta \rangle \neq 0$, by (2.2) of [6], which proves the lemma when G is exceptional.

If G is not exceptional then G is locally-isomorphic to one of the classical groups:

$$SU(l + 1), \qquad SO(2l + 1), \qquad Sp(l), \qquad SO(2l).$$

It is shown in §4 of [6] that each of

$$\sigma\pi_{2l+1}U(l + 1), \qquad \sigma\pi_{4l-1}SO(2l + 1), \qquad \sigma\pi_{4l-1}Sp(l),$$

contains elements of odd order, and it follows from (18.2) of [4] that the same is true of $\sigma\,\pi_{4l-5}SO(2l)$ $(l \neq 1)$. Furthermore

$$\pi_r SU(l + 1) \approx \pi_r U(l + 1), \qquad (r \geqq 2),$$

under the injection, and so $\sigma\,\pi_{2l+1}SU(l+1) \neq 0$. Since the Samelson product is an invariant of the structure class this completes the proof of the lemma.

To deduce the theorem we recall that a compact connected Lie group G is locally isomorphic to G', say, where G' is the direct product of an abelian group T with various nonabelian simple groups. When any of these latter are present there exists, by the lemma, some value of q such that $\sigma\,\pi_q(G') \neq 0$ and hence $\sigma\,\pi_q(G) \neq 0$. Thus $G' = T$ if G is homotopy-abelian, and hence the theorem follows at once. A maximal

[4] See (7.2) and (19.1) of [2].

compact connected subgroup of a Lie group is a deformation retract of the component of the identity [7], and so the corollary is an immediate consequence of the theorem.

REFERENCES

1. J. Adem, *Relations on iterated reduced powers*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 636–638.

2. A. Borel, *Sur la cohomologie des espaces fibrés . . .* , Ann. of Math. vol. 57 (1953) pp. 115–207.

3. ———, *Topology of Lie groups and characteristic classes*, Bull. Amer. Math. Soc. vol. 61 (1955) pp. 397–432.

4. A. Borel and J.-P. Serre, *Groupes de Lie et puissances reduites de Steenrod*, Amer. J. Math. vol. 75 (1953) pp. 409–448.

5. I. M. James, *On H-spaces and their homotopy groups*, (to be published in Oxford Quart. J. of Math.).

6. I. M. James and E. Thomas, *Which Lie groups are homotopy-abelian?* Proc. Nat. Acad. Sci. U.S.A. vol. 45 (1959) pp. 737–740.

7. H. Samelson, *Topology of Lie groups*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 2–37.

8. J.-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. vol. 58 (1953) pp. 258–294.

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