HOMOTOPY-ABELIAN LIE GROUPS

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A topological group $G$ is said to be homotopy-abelian if the commutator map of $G \times G$ into $G$ is nullhomotopic. Examples can be given of non-compact Lie groups which are homotopy-abelian but not abelian. The purpose of this note is to prove

**Theorem.** A compact connected Lie group is homotopy-abelian only if it is abelian.

**Corollary.** If a Lie group is homotopy-abelian, then its maximal compact connected subgroup is abelian.

Our proof depends on the theory of [2]. Thus we consider the Samelson "commutator" product in the homotopy groups of $G$, which is trivial when $G$ is homotopy-abelian. The product of $\alpha \in \pi_p(G)$ with $\beta \in \pi_q(G)$ is denoted by $\langle \alpha, \beta \rangle \in \pi_{p+q}(G)$, where $p, q \geq 1$. If $h$ is a homomorphism of $G$ into another topological group then

$$h_\# \langle \alpha, \beta \rangle = \langle h_\# \alpha, h_\# \beta \rangle,$$

where $h_\#$ denotes the induced homomorphism. Note that $h_\#$ is an isomorphism if $h$ is a covering map and $p, q \geq 2$. Hence if two topological groups have a common universal covering group then their higher homotopy groups are related by an isomorphism which is compatible with the Samelson product. Let $\sigma \pi_q(G)$, where $q \geq 1$, denote the subset of $\pi_{2q}(G)$ consisting of elements $\langle \beta, \beta \rangle$, where $\beta \in \pi_q(G)$. We assert the following

**Lemma.** Let $G$ be a compact connected simple non-abelian Lie group of dimension $n$ and rank $l$. Then $\sigma \pi_q(G) \neq 0$, where $q = 2n/l - 3$.

The proof is by application of (2.2) of [6]. We distinguish between the classical and exceptional cases, beginning with the latter.

Let $G$ be one of the exceptional groups. Then $n/l = p$, an odd prime number, and $G$ has no $p$-torsion (see [3]). The mod $p$ cohomology of $G$ is an exterior algebra on a basis of $l$ generators. There is one generator $y$ in dimension $q$, while the remainder are of lower dimension. It follows from Proposition 6 on page 291 of [8] that $y$ has a nontrivial

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1 Research supported in part by U.S. Air Force Contract AF 49(638)-79.
2 Such as the 2-dimensional affine group (example suggested by H. Samelson).
3 The theory of the Samelson product is given in [5], for example.

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image under the homomorphism induced by some map of $S^q$ into $G$. Thus $y$ has nonzero index, in the sense of [6], with regard to some element $\beta \in \pi_q(G)$. By Borel's theorem the mod $p$ cohomology of $B$, the classifying space of $G$, is a polynomial algebra on a basis of $l$ generators which correspond under transgression to those of the exterior algebra. The generator $x$ corresponding to $\beta$ has a nontrivial image under the homomorphism induced by some map of $S^{q+1}$ into $B$. In the polynomial algebra let $M$ denote the ideal generated by all the basis elements except $x$. If $z$ is such a generator then

$$\dim z < \dim x = q + 1 = 2(p - 1),$$

and so $\partial^s z \in M$, where $\partial^s (s \geq 0)$ denotes the Steenrod operator. Hence $\partial^s M \subseteq M$, by the Cartan product formula. This proves that $\partial^1 x \in M$, since by the Adem relation [1] we have

$$\partial^{p-2} \partial^1 x = (p - 1) \partial^{p-1} x = (p - 1) x^p \in M.$$  

Hence $\partial^1 x \equiv c x^2 \mod M$, where $c \neq 0$, and so $\partial^1 x$ is significant with regard to $\beta$, in the sense of [6]. Therefore $\langle \beta, \beta \rangle \neq 0$, by (2.2) of [6], which proves the lemma when $G$ is exceptional.

If $G$ is not exceptional then $G$ is locally-isomorphic to one of the classical groups:

$$SU(l + 1), \quad SO(2l + 1), \quad Sp(l), \quad SO(2l).$$

It is shown in §4 of [6] that each of

$$\sigma \pi_{2l+1} U(l + 1), \quad \sigma \pi_{4l-1} SO(2l + 1), \quad \sigma \pi_{4l-1} Sp(l),$$

contains elements of odd order, and it follows from (18.2) of [4] that the same is true of $\sigma \pi_{4l-5} SO(2l)$ ($l \neq 1$). Furthermore

$$\pi_r SU(l + 1) \approx \pi_r U(l + 1), \quad (r \geq 2),$$

under the injection, and so $\sigma \pi_{2l+1} SU(l+1) \neq 0$. Since the Samelson product is an invariant of the structure class this completes the proof of the lemma.

To deduce the theorem we recall that a compact connected Lie group $G$ is locally isomorphic to $G'$, say, where $G'$ is the direct product of an abelian group $T$ with various nonabelian simple groups. When any of these latter are present there exists, by the lemma, some value of $q$ such that $\sigma \pi_q(G') \neq 0$ and hence $\sigma \pi_q(G) \neq 0$. Thus $G' = T$ if $G$ is homotopy-abelian, and hence the theorem follows at once. A maximal

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\[4\] See (7.2) and (19.1) of [2].
compact connected subgroup of a Lie group is a deformation retract of the component of the identity [7], and so the corollary is an immediate consequence of the theorem.

References

5. I. M. James, On H-spaces and their homotopy groups, (to be published in Oxford Quart. J. of Math.).