LEOPOLD FEJÉR: IN MEMORIAM

1880–1959

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On October 15, 1959, Leopold Fejér died in Budapest, Hungary. For the past half century Fejér was a central figure of the international mathematical community and a leading personality of his native country. He had many pupils scattered all over the world and many friends and admirers in all countries where mathematics is at home. His profound influence was due not only to his deep and fundamental contributions to various chapters of Analysis, but also to the simplicity and elegance of his presentation, and last but not least, to his charming and suggestive personality. His loss is deplored by many mathematicians, a considerable number of them in this country. He visited the U. S. in 1933 in response to an invitation by the Century of Progress Exposition in Chicago and the American Association for the Advancement of Science. As a visiting lecturer of the Society he was the guest of more than 15 colleges and universities, all of them east of the Mississippi. Brown University conferred an honorary doctorate on him. During his long mathematical and academic career Fejér received many honors and distinctions. He was a member of the Hungarian Academy of Sciences, corresponding member of the Academies of Göttingen and München and of the Polish Academy of Sciences. He was honorary member of the Calcutta Mathematical Society, honorary doctor of the University of Budapest, and received the Kossuth prize and other distinctions in Hungary. He was a vice-president of the International Congress held in Cambridge, England, in 1912, and a member of the editorial boards of the Rendiconti del Circolo Matematico di Palermo and of the Mathematische Zeitschrift.

Fejér was born in 1880 in Pécs, Hungary. After completion of the secondary school in Pécs, he participated in the Eötvös-competition of the Hungarian Mathematical Society (initiated in 1894) and won the second prize. He studied in Budapest from 1897 until he received his Ph.D. degree in 1902 with his discovery of the Cesàro summability of Fourier series, a result which became a classical piece of Analysis. He spent the academic year 1899–1900 in Berlin where he came in contact with Hermann Amandus Schwarz. In the famous seminar conducted by this eminent mathematician he met for the first time C. Carathéodory and E. Schmidt, and later E. Landau and I. Schur, acquaintances which became the source of life long sympathies, warm friendships and fruitful scientific collaboration. Schwarz exercized a deep influence on Fejér's mathematical thinking; his interest in Fourier series and in the logarithmic potential probably dates from this time. Also the geometrical approach to mathematics and the emphasis on extremum properties became lasting effects of his contact with Schwarz. In 1902–1903 further visits followed to Göttingen and Paris, and in subsequent years Fejér was a frequent visitor in Germany, mainly in Göttingen and Berlin. His work as an academic teacher started at the University of Budapest in 1903. In 1905 he became assistant at the University of Kolozsvár (=Klausenburg=Cluj) where he collaborated with L. Schlesinger and obtained the *venia legendi*. In 1911 he was appointed full professor at the University of Budapest and remained in Hungary, except for short visits to foreign places, for the rest of his life, in spite of the cruel sufferings which the diverse political turbulences of this country inflicted on him almost continuously ever since 1914.

Fejér was active in that direction of Analysis which was prevalent in the first half of our century. His contributions to Fourier series opened up a new chapter in this field and were used in the work of such prominent mathematicians as Hurwitz, Lebesgue, de la Vallée-Poussin, Hardy, Gronwall, and Bohr. The wonderful unity of his mathematical work can be best illustrated by observing that the regular behavior of the Cesàro means (Fejér means) of the Fourier series is ultimately due to the positivity of the Fejér kernel. A similar property is the key to the summability of second order of the Laplace series, and as a matter of fact also to the regular behavior of the step parabolas which Fejér introduced in the theory of interpolation (see below). Thus this search for positive kernels (in the wider sense of the word) became a sort of *Leitmotiv* of his mathematical efforts.

Limitations of space do not permit details about Fejér's colorful and many-sided accomplishments. I restrict myself to a few particularly outstanding results. Also the Bibliography is by no means complete (Fejér published a total of 106 papers); it includes a few papers pertaining to the topics discussed here and a few other items connected with his work.

1. Cesàro summability of Fourier and Laplace series. Let $f(\theta)$ be a continuous function periodic with period 2π,

$$f(\theta) \sim a_0 + 2 \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), \qquad a_n + ib_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{int} dt,$$

its Fourier series. It is well known that the partial sums

$$
\tag{1}
\begin{aligned}
s_n(\theta) &= a_0 + 2 \sum_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin(2n+1)\dfrac{t-\theta}{2}}{\sin \dfrac{t-\theta}{2}} dt
\end{aligned}
$$

display in general a rather irregular behavior as $n \to \infty$. There exist examples of $f(\theta)$ for which at certain points the sequence $\{s_n(\theta);\ n=0, 1, 2, \cdots\}$ oscillates between $-\infty$ and $+\infty$. (Fejér himself constructed particularly simple and elegant examples of this phenomenon using a method which is applicable also in other cases; cf. [**10**, Chapter VIII].) In 1900 Fejér [**1**] discovered that the Cesàro means of the partial sums, that is the expressions

$$
\sigma_n(\theta) = \frac{s_0(\theta) + s_1(\theta) + \cdots + s_n(\theta)}{n+1},
$$

display a strikingly simpler behavior: First, for all values of θ and n they remain between the minimum and maximum of the function $f(\theta)$ and second, they tend, as $n \to \infty$, to $f(\theta)$, even uniformly for all θ. These facts are based principally on the representation

$$
\tag{2}
\sigma_n(\theta) = \frac{1}{n+1} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left\{ \frac{\sin(n+1)\dfrac{t-\theta}{2}}{\sin \dfrac{t-\theta}{2}} \right\}^2 dt
$$

in which the "Fejér kernel" appears the first time. It is non-negative in contrast to the "Dirichlet kernel" of $s_n(\theta)$ which changes sign an increasing number of times as n increases.

Let f be a continuous point function on the unit sphere, and

$$
f \sim \sum_{n=0}^{\infty} Y_n
$$

its Laplace series, that is, Y_n is a surface harmonic of degree n. Then the Cesàro means of *second* order display the same regular behavior as the Cesàro means of the first order do for the Fourier series.

Both of these results initiated an entirely new point of view in the treatment of these expansions and of similar orthogonal expansions important for Mathematical Physics. They represent a great success for the concept of summability as applied to series of functions. Concerning the numerous implications for trigonometric series we refer to the treatise of Zygmund [10]. Sturm-Liouville expansions, expansions in terms of orthogonal polynomials, and the theory of almost periodic functions have likewise profited from this idea.

2. **Interpolation in the real and complex domain.** (a) Fejér's interest in interpolation went back to 1913. The first instance of the "step parabolas" appears in the form of trigonometric interpolation as follows. In analogy to the Dirichlet integral (1) we form the trigonometric polynomial of degree n

$$S_n(\theta) = \frac{1}{2n+1}\sum_{\nu=0}^{2n} f(\theta_\nu)\,\frac{\sin(2n+1)\dfrac{\theta_\nu-\theta}{2}}{\sin\dfrac{\theta_\nu-\theta}{2}}, \qquad \theta_\nu = \frac{2\pi\nu}{2n+1},$$

characterized by the interpolation conditions $S_n(\theta_\nu) = f(\theta_\nu)$. In the same vein we form, in analogy to Fejér's integral (2), the trigonometric polynomial of degree n

$$M_n(\theta) = \frac{1}{(n+1)^2}\sum_{\nu=0}^{n} f(\theta_\nu)\left\{\frac{\sin(n+1)\dfrac{\theta_\nu-\theta}{2}}{\sin\dfrac{\theta_\nu-\theta}{2}}\right\}^2, \qquad \theta_\nu = \frac{2\pi\nu}{n+1},$$

characterized this time by the conditions $M_n(\theta_\nu) = f(\theta_\nu)$, $M_n'(\theta_\nu) = 0$. (Observe that the symbol θ_ν has a different meaning in these two cases.) If $f(\theta)$ is continuous and periodic with period 2π, the sequence $\{S_n(\theta);\, n = 0, 1, 2, \cdots\}$ might oscillate between $-\infty$ and $+\infty$. On the other hand, the quantities $M_n(\theta)$ always remain between the bounds of $f(\theta)$ and tend, as $n \to \infty$, to $f(\theta)$, uniformly for all θ. Thus we have a complete repetition of the findings of [1].

We know that Dunham Jackson formally anticipated Fejér in forming the expression $M_n(\theta)$ (cf. the paper [8], submitted on August 24, 1913). But there can be no doubt that they came independently to these important ideas: in October 1913 Fejér's results were presented by G. Pólya to the *Mathematische Gesellschaft* in Göttingen. Moreover in Fejér's work on interpolation this was only the first step. In a sequence of beautiful papers [2; 3; 4] he systematically investigated the kind of interpolation usually attributed to Hermite in which a rational polynomial y of degree $2n-1$ is sought for which y and y' have preassigned values at n given nodal points. The step parabolas are distinguished by the condition that $y' = 0$ (or more generally that the data for y' are uniformly bounded). Choosing for the nodal points, as in Gauss interpolation, the zeros of Legendre's polynomial, and assuming $y'=0$, the resulting polynomials have all the essential properties of $M_n(\theta)$ pointed out above; in this case the natural range is the interval $[-1, +1]$. From this point of view, Fejér investigated not only the polynomials associated with the zeros of ultraspherical and Jacobi polynomials, but he also discussed the general question of finding all sequences of nodal points for which the associated step parabolas display the same simple behavior as for the Gaussian abscissas. It turns out that all the fundamental polynomials must keep a constant sign in the range of interpolation, a condition which amounts again to the positivity of a certain "kernel." Fejér's work on interpolation in the real domain found a strong response in the recent work of Hungarian analysts (Erdös, Turán, Egerváry, Balázs, etc.).

In [**4**] (dated 1930) Fejér presented a very elegant proof for the following important theorem of Faber: In whatever way we prescribe a sequence of nodal points in $[-1, +1]$, there exists a continuous function such that the associated Lagrange polynomials (not the stair parabolas) are unbounded in the range $[-1, +1]$.[1]

(b) Fejér's work in the direction of complex interpolation is less voluminous than in the real domain but equally original and elegant (cf. [**5**]). Let C be a rectifiable Jordan curve in the complex z-plane. We seek a sequence of nodal points $(z_1^{(n)}, z_2^{(n)}, \cdots, z_n^{(n)}; n=1, 2, \cdots)$ situated on C such that $f(z)$ being any analytic function regular in the closed interior of C, the sequence of the associated Lagrange polynomials converges uniformly to $f(z)$ provided z is restricted to a closed domain entirely in the interior of C. Inspired by the concept of logarithmic potential, Fejér chose the nodal points in the following way. Let $x=\phi(z)$ be the function mapping the *exterior* of C onto $|x|>1$ conformally and in a schlicht manner, moreover such that $z=\infty$ is transformed in $x=\infty$ and dx/dz is real and positive as $z=\infty$. Then for each n the points $z_\nu^{(n)}$ are chosen as the images of n points $x_\nu^{(n)}$ regularly distributed on the unit circle $|x|=1$. This condition can be generalized by replacing $x_\nu^{(n)}$ by n points which are "regularly distributed" in the asymptotic sense of H. Weyl. Later Kalmár showed that this condition is not only sufficient but also necessary.

Two other remarkable results of Fejér should be mentioned.

3. **Asymptotic behavior of Laguerre polynomials.** No explicit reference is made in the pertinent publications of Fejér (cf. [**6**]) to

[1] [10] refers in this respect to a construction of Marcinkiewicz, published in 1937, which is indeed quite similar to that of Fejér (as Marcinkiewicz himself says).

Laguerre polynomials; yet they deal with the asymptotic behavior (as $n \to \infty$) of the coefficients of the power series

$$(1-z)^{-\alpha-1} e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n$$

which are indeed the Laguerre polynomials [**9**, (5.1.9)]. Fejér proves, at least in the special case $x=1$, the following important formula:

$$L_n^{(\alpha)}(x) = \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4} \cos\left(2(nx)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + R_n,$$

$$x > 0, \qquad |R_n| < An^{\alpha/2-1/2}.$$

He considers this problem in the general frame of ideas of Darboux, seeking information on the asymptotic behavior of the coefficients of a power series from knowledge of the nature of the singularities on the circle of convergence. However, in the present case the method of Darboux fails and a direct attack is necessary. The method of Fejér is very similar to that termed by Watson and others as the method of stationary phase which goes back to certain ideas of Riemann. This result of Fejér initiated a long sequence of investigations on the asymptotic behavior of Laguerre polynomials for large values of the degree (see [**9**, 8.22]).

4. **Power series which map the unit circle onto a schlicht domain.** Finally we point out a result of Fejér on conformal mapping [**7**]. Let $w=f(z)=\sum_{n=0}^{\infty} c_n z^n$ define a conformal and schlicht mapping of the unit circle $|z|<1$ onto the interior of a Jordan curve C in the w-plane. According to a fundamental result of C. Carathéodory and others the function $f(z)$ will be continuous in the closed unit circle $|z| \leqq 1$ and will map this closed domain onto the closed interior of C in a one-to-one manner. Fejér proves that the power series of $f(z)$ converges uniformly over the closed unit circle $|z| \leqq 1$.

This theorem is by no means trivial. It arises by combining two facts: (a) the convergence of $\sum n|c_n|^2$ which is equivalent to the finiteness of the inner area measure of C; (b) the summability of the power series $\sum c_n z^n$ on $|z|=1$ in the sense of Abel (or Cesàro) which is a consequence of the continuity of $f(z)$ on the boundary.

Many other interesting and important contributions of Fejér to various problems in Analysis, Algebra, Mechanics, and Elementary Geometry could be quoted and discussed. We trust that the brief selection given above will convey to the reader the style and direction of ideas of this eminent mathematician and confirm the impression that he was one of the great figures in the mathematics of the first half of the twentieth century.

Bibliography

A few of the most outstanding publications of Fejér:

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2. *Über Interpolation*, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. (1916) pp. 66–91.

3. *Über Weierstrass'sche Approximation, besonders durch Hermitesche Interpolation*, Math. Ann. vol. 102 (1930) pp. 707–725.

4. *Die Abschätzung eines Polynoms in einem Intervalle, wenn Schranken für seine Werte und ersten Ableitungswerte in einzelnen Punkten des Intervalles gegeben sind, und ihre Anwendung auf die Konvergenzfrage Hermitescher Interpolationsreihen*, Math. Z. vol. 32 (1930) pp. 426–457.

5. *Interpolation und konforme Abbildung*, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. (1918) pp. 319–331.

6. *Sur une méthode de M. Darboux*, C. R. Acad. Sci. Paris vol. 147 (1908) pp. 1040–1042.

7. *Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene*, Mathematische Abhandlungen, Hermann Amandus Schwarz gewidmet, 1914, pp. 42–53.

Other literature:

8. D. Jackson, *A formula of trigonometric interpolation*, Rend. Circ. Mat. Palermo, vol. 37 (1914) pp. 371–375.

9. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc., Colloquium Publications, vol. 23, rev. ed., 1959.

10. A. Zygmund, *Trigonometric series*, 2d ed., vols. 1 and 2, 1959.

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